

Quantifying Ambiguity in the Stock Market Using Option Data

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Abstract

I quantify the level of ambiguity in the US stock market using option data. Firstly, I set up a representative agent asset pricing model where ambiguity is modelled using the framework of recursive multiple-priors utility. This model serves as an economic foundation for a reduced-form model which keeps its main characteristics and is easy to estimate. Then, I derive a closed-form option pricing formula based on this reduced-form model of the market index. I also provide an asymptotic expansion for the corresponding short-maturity at-the-money implied volatility. I use these results to quantify the level of ambiguity with option data from 1996 to 2020. I find that the role of ambiguity is more pronounced for the recent stock market crash due to COVID-19 than for the 2008 financial crisis. Furthermore, the level of ambiguity spikes during crises and decreases to its normal level more rapidly than risk.

1 Introduction

In the field of decision-making under uncertainty, risk refers to the case that an outcome is randomly drawn from a known probability distribution, whereas ambiguity describes the situation where the outcome's distribution is unknown or non-unique. Although these two concepts are inherently different, researchers often fail to distinguish them clearly, especially in empirical studies. An example is the dual role of the famous VIX Index. The VIX is an index of 30-day option-implied volatility in the S&P 500 Index. Some studies use it as a measure of risk, such as Ang et al. (2006), Banerjee et al. (2007), and Pástor and Veronesi (2013), while others use it

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as a proxy of ambiguity, such as Bloom (2009), Baker et al. (2016), and Leduc and Liu (2016). In this paper, I develop a model allowing me to disentangle ambiguity from risk. With this model, I use option data to quantify the level of ambiguity and the implied level of risk. The results show that although the level of ambiguity has a positive correlation with the implied level of risk, they have different properties especially during crises. The level of ambiguity spikes and rapidly decreases to its normal level, while the level of implied risk remains at a high level for a longer period.

I first set up a representative agent asset pricing model in which ambiguity is modelled using the framework of recursive multiple-priors utility in Chen and Epstein (2002). Roughly speaking, a representative investor wants to maximize her utility in an infinite-horizon economy by choosing her consumption level and allocating her wealth between a risk-free asset and a risky asset. There is a model of the aggregate dividend process, but unfortunately she does not know whether it is the true model. Instead, she has a set of alternative models in mind. Each candidate model in this set is represented by a probability measure, which is defined by the Radon-Nikodym derivative (i.e., likelihood ratio) with respect to a reference measure. Under different measures in the alternative set, the dividend process has the same diffusion term which serves as the source of risk. The level of ambiguity is determined by the size of the alternative set which is the boundary of the density generator of the Radon-Nikodym derivatives. When the bound is large (respectively, small), the alternative set is large (respectively, small), and thus the investor faces a high (respectively, low) level of ambiguity. By Girsanov theorem, this bound affects the drift term of the dividend process under reference measure. I further assume the level of ambiguity is not constant but time-varying, and follows a stochastic process. With this assumption, I allow for the fluctuation of the ambiguity level.

I solve the model in equilibrium, and obtain the price of the risky asset and the risk-free rate. The risk-free rate is a function of ambiguity. Facing high ambiguity, the investor is willing to accept a low risk-free rate to hold the safe asset. The equilibrium equity premium is made up of three parts. The first part is the compensation for bearing risk. The latter two parts are the compensations for bearing ambiguity, with one for the current level of ambiguity and the other for the future fluctuations of ambiguity. For the price of the risky asset, I have its dynamics not only under reference measure but also under risk-neutral measure. Its drift term is the risk-free rate under risk-neutral measure, and it has two diffusion terms with one originating from the fluctuation of consumption and the other coming from the fluctuation of

ambiguity.

This economic model is difficult to estimate directly since not all parameters in it can be identified. I replace the combination of parameters in the economic model with a single parameter, and obtain a reduced-form model for the price of the risky asset under risk-neutral measure. This reduced-form model keeps the main characteristics of the economic model, and does not affect the estimation of the ambiguity level which is the main target of this paper.

Based on the reduced-form model, I derive a closed-form pricing formula for European call options on the risky asset. The method being used comes from Duffie et al. (2000) who provide a general solution to the option pricing problem for affine diffusions. This option pricing formula exhibits a relation between the ambiguity level and option prices, but this relation is not that straightforward. To study the impact of the ambiguity level on options directly, I further derive an asymptotic expansion for the implied volatility¹. Using the perturbation method in Jacquier and Lorig (2015), I obtain the asymptotic expansion for the implied volatility as a function of the natural logarithm of moneyness² up to the second order, in the short-maturity limit, and around the at-the-money level. The at-the-money level of implied volatility is determined by the sum of risk and ambiguity, while the at-the-money slope³ is affected by the fraction of ambiguity in their sum.

These theoretical results under the reduced-form model are used to quantify the level of ambiguity and the level of implied risk. The asymptotic expansion offers a possibility of identifying risk and ambiguity from the level and slope of the implied volatility curve around the at-the-money level. Although the results obtained by this method are approximations, it serves as a quick way to learn about the level of risk and ambiguity in the market. The formal estimation strategy is based on the option pricing formula for the reduced-form model. In this paper, I make use of the two-step calibration method in Christoffersen et al. (2009), which views the ambiguity level as an extra parameter that needs to be estimated. The basic idea is as follows. On the one hand, there are different option contracts with known prices in the market. On the other hand, given the parameter values, I can calculate each option's price using the option pricing formula derived previously. In principle, the level of ambiguity is estimated by making the model-based option prices as close

¹ Implied volatility is more commonly used as a measure of an option's value than its price.

² Moneyness is defined as the strike price of the option contract over the current price of the underlying asset.

³ To be precise, the slope here is the first order derivative of implied volatility with respect to the logarithm of moneyness

as possible to the market prices. Since investors typically use implied volatilities to represent option prices in practice, I follow this tradition and compare implied volatilities instead of option prices to obtain the estimation of the ambiguity level. The main dataset used in this paper consists of options written on the S&P 500 Index, ranging from January 1996 to December 2020. It covers three recessions in the United States determined by NBER⁴: the bursting of the dot-com bubble, the financial crisis of 2008-2009, and the most recent stock market crash due to COVID-19.

My main results are as follows. Firstly, the level of ambiguity is on average lower than the level of implied risk, but it is more volatile. Their sum measures the total variance of the market return, which is comparable with the squared VIX Index. This fact explains the dual role of the VIX Index in empirical studies. In fact, the VIX contains information about both risk and ambiguity, and with the model in this paper I can separate ambiguity from risk. The second finding is about recessions, especially the 2008-2009 financial crisis and the recent COVID-19 stock market crash. During both periods, the level of ambiguity and the level of implied risk are much higher than usual. Yet, these two recessions are driven by different sources. The 2008 recession has balanced levels of ambiguity and implied risk whose average levels are quite similar, while the 2020 recession is mainly driven by ambiguity with the level of ambiguity dominating the level of implied risk. Thirdly, the level of ambiguity decreases to the normal level quickly after reaching peaks during crises, but the level of implied risk remains at a high level for a longer period.

This paper is mainly related to two branches of the literature. The first one is the measurement of ambiguity. Since ambiguity describes the situation where the distribution of the outcome is unknown or non-unique, it is difficult to be measured directly in practice. A number of methods that quantify the ambiguity level have been developed, including the implied volatilities in the stock market (Bloom, 2009), the forecast disagreement on macroeconomic and financial variables (Anderson et al., 2009; Bachmann et al., 2013), the newspaper coverage frequency of economic policy uncertainty (Baker et al., 2016), the volatility of expected volatility for stock returns (Baltussen et al., 2018), the function of statistical quantities such as likelihood ratio tests and Type I and Type II errors (Balter and Pelsser, 2021), and the expected volatility of probabilities (Izhakian, 2020). However, without a solid theoretical foundation, most of these measures tend to either mix up ambiguity with other information or capture only partial information about ambiguity. VIX, the

⁴ Please refer to <https://www.nber.org/research/business-cycle-dating>.

implied volatility of the S&P 500 Index, combines information about stock market risk with ambiguity. The degree of forecast disagreement measures a dispersion of beliefs among forecasters rather than the ambiguity about macroeconomic and financial variables. The Economic Policy Uncertainty Index reflects the frequency of newspaper articles with terms about "economics", "policy", and "uncertainty". Yet, the usage of "uncertain" in non-academic writing is flexible and may contain the meaning of both risk and ambiguity. The volatility of expected volatility only considers the variance of the second moment of return distributions, and excludes the information about ambiguity embedded in the higher moments. Izhakian (2020) proposes a measure of ambiguity with a theoretical foundation. The framework he uses is the expected utility with uncertain probabilities which is introduced in Izhakian (2017). However, in Fu et al. (2021) we find that his ambiguity measurement $\mathcal{U}^2$ is not an equivalent way of representing preferences it is supposed to represent. My paper contributes to this literature by providing a new method to quantify the daily level of ambiguity using option data based on the framework of recursive multiple-priors utility.

Second, this paper relates to the literature that analyzes the effect of ambiguity on option prices theoretically. Despite using different models of ambiguity⁵, a number of theoretical models have been developed to study ambiguity in the context of asset pricing. Most of them focus on the equity market (Chen and Epstein, 2002; Trojani and Vanini, 2002, 2004; Maenhout, 2004; Sbuelz and Trojani, 2008; Ju and Miao, 2012; Jeong et al., 2015) and the bond market (Garlappi et al., 2007; Ulrich, 2013; Shi, 2019; Zhao, 2020). The studies of ambiguity and options are relatively few. Liu et al. (2005) are among the first to use ambiguity to explain option prices in equilibrium. They assume investors are uncertain about rare events and use the framework of robust control to explain the volatility skewness of index options. Drechsler (2013) extends their work by separating the volatility of consumption from the volatility of returns and explains the variance premium as well as the volatility skewness. Miao et al. (2019) also provide an ambiguity-based interpretation of the variance premium, but the framework they use is the smooth ambiguity model. The model I set up in this paper provides implications on both the market index and its options. The equity premium contains the compensations for bearing not only risk but also ambiguity, which is consistent with the previous literature about the effect of ambiguity on the equity premium. Furthermore, I derive a closed-form option

⁵ The possible frameworks are: the robust control framework of Hansen and Sargent (2001), the multiple-priors utility framework of Gilboa and Schmeidler (1989), and the smooth ambiguity framework of Klibanoff et al. (2005).

pricing formula for the market index, which has not been available in the previous literature up to my knowledge. I also provide an asymptotic expansion for the implied volatility in the short-maturity limit and around the at-the-money level. This approximation exhibits clearly how the interplay between risk and ambiguity affects the shape of the implied volatility curve.

The rest of this paper is organized as follows. In Section 2, I first set up an economic model and solve for its equilibrium. Then I write down its reduced-form version and derive the option pricing formula based on it. I also present the asymptotic expansion for implied volatility under the reduced-form model. Section 3 describes the data. Section 4 presents the estimation method and the empirical results. Section 5 is the conclusion.

2 Model

In this section, I first set up an economic model, which is a representative agent asset pricing model under the framework of recursive multiple-priors utility following Chen and Epstein (2002). The level of ambiguity is allowed to be time-varying in this model. Based on this economic model, I write down a reduced-form model which keeps the main characteristics of the economic model and makes the later estimation easier. In other words, this economic model serves as an economic foundation of the reduced-form model. Under the reduced-form model, I derive a closed-form option pricing formula. I also provide an asymptotic expansion for the implied volatility around the at-the-money level. Both of these two results will be used in Section 4 to quantify the level of ambiguity in the US stock market with option data.

2.1 Economic Model

The setting for the economic model is an infinite-horizon, continuous-time economy with a representative investor. This investor has a reference model of the economy in mind, but is uncertain whether it is the true model. Instead, she has a set of alternative models, and makes consumption and portfolio choice decisions to maximize her utility across these alternative models.

Modeling Ambiguity Aversion

Consider a probability space $(\Omega, \mathcal{F}, P_0)$, where B_t is a d -dimensional Brownian motion defined on it. Decision-maker is uncertain about whether P_0 is a true model⁶, and thus takes P_0 as the reference model rather the true model. An alternative model P is defined with respect to the reference model P_0 using the Radon-Nikodym derivative:

$$\left. \frac{dP}{dP_0} \right|_t = z_t,$$

where

$$dz_t = -z_t \theta_t dB_t.$$

θ_t is a density generator and belongs to the set Θ . Since a density generator in Θ corresponds to an alternative model P , the set of alternative models P^Θ is built up corresponding to Θ .

To model preferences in the presence of ambiguity, I adopt the framework of recursive multiple-priors utility:

$$V_t = \min_{P \in P^\Theta} E^P \left[\int_t^\infty f(C_s, V_s) ds | \mathcal{F}_t \right], \quad (1)$$

where $\{C_t | t \in [0, \infty)\}$ are consumption streams. For the choice of the aggregator function f , I use Epstein-Zin preferences to allow for the separation of elasticity of intertemporal substitution (EIS) and risk aversion:

$$f(C, V) = \beta \frac{\alpha}{\rho} V \left[\frac{C^\rho}{(\alpha V)^{\rho/\alpha}} - 1 \right],$$

where $\beta \geq 0$ is the rate of time preference, $\rho \in [0, 1]$ is EIS and $\alpha \in [0, 1]$ is the risk attitude.

By Girsanov theorem, a change of measure from P_0 to any $P \in P^\Theta$ affects only the drift term of the utility continuation process. With the multiple-priors utility, the decision-maker acts following the worst-case model. According to Chen and Epstein (2002) and Shi (2019),

$$dV_t = [-f(C_t, V_t) + \max_{\theta \in \Theta} \theta_t \cdot \sigma_{V,t}] dt + \sigma_{V,t} dB_t.$$

⁶ A model is characterized by its probability measure.

Set of Alternative Models

I consider the case where $B_t = (B_{1,t}, B_{2,t})'$ is of two dimensions and $B_{1,t}$ is uncorrelated with $B_{2,t}$. As a result, the corresponding density generator θ_t is also of two dimensions and written as $(\theta_{1,t}, \theta_{2,t})'$.

The set of density generators is assumed to be

$$\Theta = \{(\theta_{1,t}, \theta_{2,t})' : -\kappa_t \leq \theta_{1,t} \leq \kappa_t, \theta_{2,t} = 0, \forall t \in [0, +\infty)\}.$$

Since ambiguity concerns whether B_t is a Brownian motion under P_0 ⁷, the above assumption of Θ indicates that there is no ambiguity about $B_{2,t}$ and it is always a Brownian motion under P_0 . On the contrary, there is ambiguity about $B_{1,t}$ and $\theta_{1,t}$ is bounded by κ_t .

In fact, κ_t determines the size of the alternative set P^Θ . If κ_t is large, the set of alternative models is also large, and the decision-maker faces a larger uncertainty when making decisions. This is why κ_t is interpreted as the level of ambiguity. κ_t can be constant or time-varying. In this paper, I follow Shi (2019) and assume the dynamics of κ_t is a Cox-Ingersoll-Ross (CIR) process⁸:

$$d\kappa_t = a(\bar{\kappa} - \kappa_t)dt + b\sqrt{\kappa_t}dB_{2,t}. \quad (2)$$

$\bar{\kappa}$ is the long-term mean of κ_t , and a is the speed of adjustment to this mean. Both of them are strictly positive to avoid the possibility that κ_t falls below zero, but κ_t can still be equal to 0 occasionally. This setting indicates that ambiguity can neither be negative nor be completely eliminated forever.

Equilibrium Asset Prices

Consider an infinite-horizon utility optimization problem with a representative investor. The aggregate dividend process is given by

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_{1,t}, \quad (3)$$

where both μ_D and σ_D are constants. $B_{1,t}$ has already been specified in the previous section. To be precise, the representative investor is uncertain about whether $B_{1,t}$

⁷ This interpretation comes from page 1415 of Chen and Epstein (2002).

⁸ This assumption follows Shi (2019), who argues that many models studying the endogenous learning process of ambiguity result in a mean-reverting process for the ambiguity level and chooses a CIR process to model the dynamics of the ambiguity level.

is a Brownian motion under reference measure P_0 . This ambiguity is bounded by κ_t , which follows (2).⁹

There is a risk-free asset and a single risky asset in this economy. The risk-free asset has an interest rate of r_t , while the risky asset is a claim on the stream of aggregate dividends D_t which has an ex-dividend price of S_t .¹⁰ The investor makes portfolio choices between these two assets. Let ω_t be the fraction of wealth W_t invested in the risky asset. The dynamics of W_t is

$$dW_t = (1 - \omega_t)W_t r_t dt + \omega_t W_t \frac{dS_t + D_t dt}{S_t} - C_t dt.$$

The first two terms are the investor's "income" from the portfolio, while the last term is her "expenditure" on the consumption good.

Given this budget constraint, the investor chooses her consumption C_t and the investment plan ω_t to optimize her utility. Let J_t be the indirect utility function of the investor,

$$J_t = \max_{(C, \omega)} \min_{P \in P^\Theta} E^P \left[\int_t^\infty f(C_s, J_s) ds \mid \mathcal{F}_t \right], \quad (4)$$

where the inner minimization is given by (1).¹¹

According to Duffie and Epstein (1992), the Hamilton-Jacobi-Bellman (HJB) equation for J is¹²:

$$\sup_{(C, \omega)} \mathcal{D}J(W, \kappa) + f(C, J(W, \kappa)) - \kappa \sigma_{J1} = 0. \quad (5)$$

(W, κ) are the two state variables that the investor faces when making decisions, and (C, ω) are control variables. $\mathcal{D}$ is the standard Dynkin differential operator.

For this HJB equation, I take the derivatives of its left-hand side with respect to C and ω , and let them equal to zero. The optimal indirect utility function $J(W, \kappa)$ should satisfy these two equations as well as the HJB equation (5). I conjecture that the function form of $J(W, \kappa)$ is

$$J(W, \kappa) = \exp \left(\frac{\alpha(1 - \rho)}{\rho} h(\kappa) \right) \frac{W^\alpha}{\alpha},$$

where $h(\kappa)$ is a function of κ .

In this economy, market clearing implies that the representative investor consumes

⁹ There is no ambiguity about the level of ambiguity itself.

¹⁰ Both r_t and S_t are determined endogenously in equilibrium.

¹¹ This equation is same as Equation (6) in Shi (2019).

¹² Relevant equations in Duffie and Epstein (1992) are Equation (13), (14), and (15). The subscript t is ignored for simplicity of notation.

exactly the aggregate dividends given by (3), so that $C^* = D$. Furthermore, in an equilibrium of a representative-agent model, there is zero net supply of the risk-free asset, i.e. $\omega^* = 1$. Substituting these two conditions into the above HJB equation, I obtain

$$0 = \mu_D - \frac{\beta}{\rho} - \frac{1}{2}(1 - \alpha)\sigma_D^2 - \kappa_t\sigma_D + \frac{1}{\rho}h'(\kappa_t)a(\bar{\kappa} - \kappa_t) + \frac{1}{2\rho}h''(\kappa_t)b^2\kappa_t + \frac{\alpha}{2\rho^2}h'(\kappa_t)^2b^2\kappa_t + \frac{1}{\rho}\beta^{1/(1-\rho)}\exp(-h(\kappa_t)). \quad (6)$$

To solve the model analytically, I use the log-linear approximation proposed by Collin-Dufresne and Goldstein (2005) and also used by Shi (2019). $h(\kappa)$ is approximated by

$$h(\kappa) \approx \eta_0 + \eta_1\kappa, \quad (7)$$

where η_0 and η_1 are obtained by minimizing the expected squared approximation error of (6), and are indicated by $\hat{\eta}_0$ and $\hat{\eta}_1$, respectively.

In this way, I derive the price of the risky asset under reference measure P_0 in equilibrium, which is presented in Proposition 1.

Proposition 1. *Up to the accuracy of approximation (7), the price of the risky asset under reference measure P_0 has dynamics*

$$\frac{dS_t}{S_t} = (r_t + \lambda_t - \frac{D_t}{S_t})dt + \sigma_D dB_{1,t} + \hat{\eta}_1 b \sqrt{\kappa_t} dB_{2,t}, \quad (8)$$

where the risk-free rate r_t , the excess return λ_t , and the dividend yield $\frac{D_t}{S_t}$ are respectively given by

$$r_t = [\beta + (1 - \rho)\mu_D - \frac{1}{2}(1 - \alpha)(2 - \rho)\sigma_D^2] - [(1 - \rho)\sigma_D + \frac{1}{2}\frac{\rho - \alpha}{\rho}\hat{\eta}_1^2 b^2]\kappa_t, \quad (9)$$

$$\lambda_t = (1 - \alpha)\sigma_D^2 + \kappa_t\sigma_D + \frac{\rho - \alpha}{\rho}\hat{\eta}_1^2 b^2\kappa_t, \quad (10)$$

$$\frac{D_t}{S_t} = [\beta + \frac{1}{2}\rho(1 - \alpha)\sigma_D^2 - \rho\mu_D - \hat{\eta}_1 a \bar{\kappa}] + [\rho\sigma_D + \hat{\eta}_1 a - \frac{\alpha}{2\rho}\hat{\eta}_1^2 b^2]\kappa_t. \quad (11)$$

Furthermore, κ_t follows (2) under reference measure P_0 .

Proof. See Appendix A. □

λ_t is the equilibrium equity premium. Its first term is the compensation for bearing the volatility of dividends. If there is no model uncertainty, this term is the entire equity premium. The second term is due to the current level of ambiguity. The

investor lacks confidence in the model of dividends. She dislikes such uncertainty and needs larger compensation for higher level of ambiguity. Besides the current ambiguity level, the fluctuations in the future level of ambiguity earn a separate premium as well, which is reflected in the last term of (10). The existence of this term is closely related to the separation between EIS and risk aversion. The investor gets an extra compensation if she prefers an earlier resolution of fluctuations in the level of ambiguity ($\rho > \alpha$).

Besides the price of the risky asset, I also derive the pricing kernel under reference measure using Lemma 5.1 of Chen and Epstein (2002). The result is presented in Proposition 2.

Proposition 2. *Up to the accuracy of approximation (7), the pricing kernel under reference measure P_0 has dynamics*

$$\frac{dM_t}{M_t} = -r_t dt - [(1 - \alpha)\sigma_D + \kappa_t]dB_{1,t} - \frac{\rho - \alpha}{\rho}\hat{\eta}_1 b\sqrt{\kappa_t}dB_{2,t}, \quad (12)$$

where r_t is the risk-free rate given by (9).

Proof. See Appendix A. □

With the pricing kernel known, I use the Girsanov theorem to change the probability measure from reference measure P_0 to risk-neutral measure Q . For the price of the risky asset, its drift term, as expected, turns to be the risk-free rate minus the dividend yield. Its diffusion terms keep unchanged with one originating from the fluctuation of dividends and the other coming from the fluctuation of ambiguity. Proposition 3 shows the result.

Proposition 3. *Up to the accuracy of approximation (7), the price of the risky asset under risk-neutral measure Q has dynamics*

$$\frac{dS_t}{S_t} = (r_t - \frac{D_t}{S_t})dt + \sigma_D \tilde{B}_{1,t} + \hat{\eta}_1 b\sqrt{\kappa_t}d\tilde{B}_{2,t}, \quad (13)$$

where the risk-free rate r_t and the dividend yield $\frac{D_t}{S_t}$ are given by (9) and (11) respectively. Furthermore, κ_t under risk-neutral measure Q follows

$$d\kappa_t = [a\bar{\kappa} - (a + \frac{\rho - \alpha}{\rho}\hat{\eta}_1 b^2)\kappa_t]dt + b\sqrt{\kappa_t}d\tilde{B}_{2,t}. \quad (14)$$

$\tilde{B}_{1,t}$ and $\tilde{B}_{2,t}$ are two standard Brownian motions under Q and their correlation is zero.

Proof. See Appendix A. □

2.2 Reduced-Form Model

The economic model developed in Section 2.1 is difficult to estimate directly since not all parameters can be identified. In this section, I write down a reduced-form model for the price of the risky asset under risk-neutral measure. This reduced-form model keeps the main characteristics of the result in Proposition 3, and does not affect the estimation of the ambiguity level which is the main target of this paper.

By replacing the combination of parameters in Proposition 3 with a single parameter, I write down the price of the risky asset parsimoniously as follows: under risk-neutral measure Q ,

$$\frac{dS_t}{S_t} = (r_t - q_t)dt + \Sigma_1 d\tilde{B}_{1,t} + \Sigma_2 \sqrt{\kappa_t} d\tilde{B}_{2,t}, \quad (15)$$

$$d\kappa_t = (\Delta_0 - \Delta_1 \kappa_t)dt + \sigma \sqrt{\kappa_t} d\tilde{B}_{2,t}, \quad (16)$$

$$r_t = \Lambda_0 - \Lambda_1 \kappa_t, \quad (17)$$

$$q_t = \zeta_0 + \zeta_1 \kappa_t, \quad (18)$$

where

$$\left\{ \begin{array}{l} \Lambda_0 = \beta + (1 - \rho)\mu_D - \frac{1}{2}(1 - \alpha)(2 - \rho)\sigma_D^2 \\ \Lambda_1 = (1 - \rho)\sigma_D + \frac{1}{2}\frac{\rho - \alpha}{\rho}\hat{\eta}_1^2 b^2 \\ \zeta_0 = \beta + \frac{1}{2}\rho(1 - \alpha)\sigma_D^2 - \rho\mu_D - \hat{\eta}_1 a \bar{\kappa} \\ \zeta_1 = \rho\sigma_D + \hat{\eta}_1 a - \frac{\alpha}{2\rho}\hat{\eta}_1^2 b^2 \\ \Delta_0 = a \bar{\kappa} \\ \Delta_1 = a + \frac{\rho - \alpha}{\rho}\hat{\eta}_1 b^2 \\ \Sigma_1 = \sigma_D \\ \Sigma_2 = \hat{\eta}_1 b \\ \sigma = b \end{array} \right. .$$

In this reduced-form model, the price of the risky asset is still driven by both the fluctuations of dividends and the fluctuations of ambiguity, similar as in Proposition 3. Σ_1 is the volatility of dividends which represents the level of risk, while κ_t is the level of ambiguity. The risk-free rate r_t and the dividend yield q_t are still affine functions of the ambiguity level, which are similar as in the economic model.

Option Pricing Formula

Based on the reduced-form model, I derive a closed-form pricing formula for European call options on the risky asset¹³. The method being used comes from Duffie et al. (2000) who provide a general solution to the option pricing problem for affine diffusions. The result is shown in Proposition 4.

Proposition 4. *At $t = 0$, the price of the risky asset is S_0 and the level of ambiguity is κ_0 . For an European call option with strike price K and maturity date T , its price is*

$$C_0 = \left[\frac{1}{2} \psi(1; T, S_0, \kappa_0) + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re} \left(\frac{K^{-i\phi} \psi(1 + i\phi; T, S_0, \kappa_0)}{i\phi} \right) d\phi \right] - K \left[\frac{1}{2} \psi(0; T, S_0, \kappa_0) + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re} \left(\frac{K^{-i\phi} \psi(i\phi; T, S_0, \kappa_0)}{i\phi} \right) d\phi \right], \quad (19)$$

where

$$\psi(u; T, S_0, \kappa_0) = \exp \left(\bar{\alpha}(u; 0, T) + \bar{\beta}_1(u; 0, T) \ln(S_0) + \bar{\beta}_2(u; 0, T) \kappa_0 \right),$$

with

$$\begin{aligned} \bar{\beta}_1(u; 0, T) &= u, \\ \bar{\beta}_2(u; 0, T) &= -\frac{\Gamma_1(u)}{\sigma^2} + \frac{\xi(u)}{\sigma^2} \cdot \frac{(\xi(u) + \Gamma_1(u))e^{-\xi(u)T} - (\xi(u) - \Gamma_1(u))}{(\xi(u) + \Gamma_1(u))e^{-\xi(u)T} + (\xi(u) - \Gamma_1(u))}, \\ \bar{\alpha}(u; 0, T) &= -\left[\Lambda_0 - \left(\Lambda_0 - \zeta_0 - \frac{1}{2} \Sigma_1^2 \right) u - \frac{1}{2} \Sigma_1^2 u^2 \right] T \\ &\quad - \Delta_0 \left[\frac{\xi(u) + \Gamma_1(u)}{\sigma^2} T + \frac{2}{\sigma^2} \ln \left(1 - \frac{(\xi(u) + \Gamma_1(u))}{2\xi(u)} (1 - e^{-\xi(u)T}) \right) \right], \end{aligned}$$

and

$$\begin{cases} \Gamma_0(u) = -2\Lambda_1 + 2(\Lambda_1 + \zeta_1 + \frac{1}{2}\Sigma_2^2)u - \Sigma_2^2 u^2 \\ \Gamma_1(u) = \Sigma_2 \sigma u - \Delta_1 \\ \xi(u) = \sqrt{\Gamma_1(u)^2 + \Gamma_0(u)\sigma^2} \end{cases}.$$

Proof. See Appendix C. □

If there is no ambiguity, the result in Proposition 4 turns to the classic Black-Scholes formula with dividends for option pricing. In Section 4, this formula will be used

¹³ I could also derive a closed-form pricing formula for European put options in a similar way, but I do not include it in this paper since it will not be used in later sections.

to estimate the level of ambiguity in the US stock market using option data on the S&P 500 Index.

Asymptotic Expansion for Implied Volatility

The option pricing formula in Proposition 4 provides a relation between the ambiguity level and option prices. However, this relation is not very straightforward due to the complex formula form. As for implied volatility, which is commonly used by investors in practice, it is even more difficult to study the impact of ambiguity on it, since implied volatility is typically obtained by some root-finding algorithms numerically rather than through analytical transformations from option prices.

To study the impact of the ambiguity level on options directly, I derive an asymptotic expansion for implied volatility under the reduced-form model using the perturbation method in Jacquier and Lorig (2015). More specifically, I focus on a special case of the reduced-form model (15) - (18), where $\Lambda_0 = 0$, $\Lambda_1 = 0$, $\zeta_0 = 0$, $\zeta_1 = 0$ and $\Sigma_2 = -1$. In this case, both the risk-free rate r_t and the dividend yield q_t are assumed to be zero.¹⁴

Jacquier and Lorig (2015) provides a method to derive the asymptotic expansion for implied volatility using the characteristic function of risky asset's log price. The basic idea is to write the characteristic function of $\ln(S_T)$ under the reference model as a perturbation around the Black-Scholes characteristic function. As a result, the call option price can be expressed as a perturbation around the Black-Scholes call price, which can then be turned into an expansion for the corresponding implied volatility. I make use of this perturbation method and derive an asymptotic expansion for implied volatility with respect to the natural logarithm of moneyness up to the second order, in the short-maturity limit¹⁵, and around the at-the-money level.

Denoting the natural logarithm of moneyness by m , the asymptotic expansion for implied volatility at time $t = 0$ is expressed as follows:

$$IV_t(m) \approx b_0 + b_1 m + b_2 m^2, \quad (20)$$

¹⁴ The assumption of $r_t = 0$ is partly supported by the worldwide low interest rate environment since 2009, and will be lifted later.

¹⁵ In other words, this approximation holds when the time to maturity is close to zero. That is also why the maturity date does not appear in the following approximation formula.

where

$$\begin{cases} b_0 &= \sqrt{\Sigma_1^2 + \kappa_0} \\ b_1 &= -\frac{1}{4} \frac{\sigma \kappa_0}{(\sqrt{\Sigma_1^2 + \kappa_0})^3} \\ b_2 &= \frac{1}{24} \frac{\sigma^2}{(\sqrt{\Sigma_1^2 + \kappa_0})^3} - \frac{5}{48} \frac{\sigma^2 \kappa_0^2}{(\sqrt{\Sigma_1^2 + \kappa_0})^7} \end{cases}.$$

In other words, implied volatility can be approximated by a quadratic function of m . I present the derivation of this approximation and discuss its accuracy in Appendix D.

According to (20), when $m = 0$ (i.e., at the money), the level of implied volatility is b_0 , which is the square root of the sum of risk and ambiguity. The at-the-money slope of $IV(m)$ is b_1 , which is affected by the relative levels of risk and ambiguity. Equation (20) offers an opportunity to figure out κ_0 and the implied values of Σ_1 and σ from its coefficients b_0 , b_1 , and b_2 . The at-the-money level of implied volatility is determined by the sum of risk and ambiguity, while the at-the-money slope¹⁶ is affected by the fraction of ambiguity in their sum.

3 Data

To quantify the level of ambiguity in the US stock market, I use option data on the S&P 500 Index from January 1996 to December 2020, which covers three recessions in the United States determined by NBER. The first one ranges from March 2001 to November 2001, which follows the bursting of the dot-com bubble. The second one is the financial crisis in 2008 which was triggered by a large scale of defaults of subprime mortgages. It starts in December 2007 and ends in June 2009. The last one is the stock market crash due to COVID-19, which starts in February 2020 and lasts for three months. The data are taken from the OptionMetrics database.

According to Figlewski (2008), DeMiguel et al. (2013), and Andreou et al. (2020), I apply the following filtering rules to the option data set. I first discard those options with no trading volume or open interest. Second, I only keep options with time to maturity larger than or equal to seven days. Finally, I remove those options that violate no-arbitrage condition obviously, which include: options with negative bid or ask prices, options whose bid price is larger than its ask price, and options whose bid or ask prices are greater than the current price of the underlying asset. Call options whose prices are smaller than $\max\{0, S - Ke^{-rT}\}$ and put options whose

¹⁶ To be precise, the slope here is the first order derivative of implied volatility with respect to the logarithm of moneyness

prices are smaller than $\max\{0, Ke^{rT} - S\}$ are also removed due to the violation of no-arbitrage condition, where S , K , T , and r are the underlying asset price, the option strike price, the option time-to-maturity, and the risk-free rate, respectively.

After filtering, I calculate the implied volatility for each option contract based on the Black-Scholes formula.¹⁷ For implied volatilities with the same maturity on the same date, I interpolate them with respect to moneyness following Figlewski (2008). A fourth-order spline with a single knot placed at the money is applied to the interpolation of implied volatilities. The spline function of implied volatility with respect to moneyness can be written as follows¹⁸:

$$IV(M) = \beta_0 + \beta_1 M + \beta_2 M^2 + \beta_3 M^3 + \beta_4 M^4 + \beta_5 \max\{M - 1, 0\}^4, \quad (21)$$

The last term, $\max\{M - 1, 0\}^4$, is added to place the single knot at the money.

For each combination of quote date and maturity date, I denote the data set of moneyness and implied volatility by $\{(M_i, IV_i)\}_{i=1}^I$, where IV_i has three components: IV based on mid price IV_i^{mid} , IV based on ask price IV_i^{ask} , and IV based on bid price IV_i^{bid} . I estimate parameters $\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5)'$ by minimizing the squared difference between $IV(M_i)$ and IV_i^{mid} . The objective function is:

$$\min_{\beta} \sum_{i=1}^I w_i (IV_i^{mid} - IV(M_i))^2.$$

Here I use the weighting function proposed by Figlewski (2008)¹⁹:

$$w_i = \begin{cases} F_{N(0,1)}(IV(M_i) - IV_i^{ask}) & \text{if } IV(M_i) \geq IV_i^{mid} \\ F_{N(0,1)}(IV_i^{bid} - IV(M_i)) & \text{if } IV(M_i) \leq IV_i^{mid} \end{cases}$$

where $IV(M_i)$ is determined by equation (21) and depends on the coefficients β . $F_{N(0,1)}(\cdot)$ is the cumulative distribution function of the standard normal distribution $N(0, 1)$.

These interpolated curves of implied volatility will be used in Section 4.

¹⁷ Here, I make use of the annualized dividend yields and the zero curves provided by Option-Metrics.

¹⁸ This expression is based on Shalizi (2013).

¹⁹ Alternatively, one could give equal weight to every data point by letting $w_i = \frac{1}{I}$, but in this way the spread of IV_i^{bid} and IV_i^{ask} is not taken into consideration. Suppose there are two points with the same distance from the mid-price IV, but one is located within the quoted spread while the other is outside. It makes sense to punish the one outside the spread more than the one within the spread, since the former one corresponds to an option contract whose price is beyond the observed bid or ask prices.

4 Estimation Results

Under the reduced-form model in Section 2.2, I have derived a close-form option pricing formula and an asymptotic expansion for implied volatility. In this section, I will use both these two results to quantify the level of ambiguity and the level of implied risk in the US stock market with S&P 500 Index options. I first focus on the asymptotic expansion for implied volatility. For a special case under the reduced-form model, the asymptotic expansion in (20) presents how the level of ambiguity and the level of implied risk affect the shape of implied volatility curve in the short-maturity limit. Based on this relation, I can identify risk and ambiguity from the level and slope of implied volatility curve around the at-the-money level. Although the results obtained by this method are approximations, it serves as a quick way to learn about the level of risk and ambiguity in the market and can be implemented even in real time. The formal estimation strategy is based on the option pricing formula in Proposition 4. Here I make use of the two-step calibration method in Christoffersen et al. (2009). The main idea is to estimate the ambiguity level and other parameter values of (19) by making the model-based implied volatilities as close as possible to the market-based ones.

4.1 Results Based on Asymptotic Expansion

In Section 2.2 under the reduced-form model, I have derived an asymptotic expansion for implied volatility with respect to the natural logarithm of moneyness up to the second order, in the short-maturity limit, and around the at-the-money level. According to (20), the at-the-money level of this implied volatility curve is the square root of the sum of risk and ambiguity, while the at-the-money slope is affected by the fraction of ambiguity in this sum. Based on this asymptotic expansion, I can learn about the ambiguity level and the implied risk level from the derivatives of implied volatility with respect to moneyness around the at-the-money level. This idea is applied to the S&P 500 Index options processed in Section 3. For each day, I choose the interpolated curve of implied volatility with the shortest maturity and calculate its derivatives with respect to moneyness up to the second order around the at-the-money level. These derivatives are used to figure out the level of ambiguity κ_0 and the implied level of risk Σ_1^2 based on (20).

Figure 1 plots daily levels of ambiguity κ_0 and risk which is represented by the implied level of Σ_1^2 from January 1996 to December 2020. The black solid line is the

level of ambiguity, while the red dashed line is the implied level of risk. The shaded areas are three recessions in the United States during 1996-2020 according to NBER, which are the dot-com bubble, the 2008 financial crisis, and the COVID-19 crash respectively. I also compare the risk and ambiguity levels with the total variance of the market return, which in this case is measured by the square of the well-known VIX Index from the CBOE. In Figure 2, I plot the square of daily VIX from 1996 and 2020. The summary statistics of risk, ambiguity, and VIX square are presented in Table 1.

During the whole period, the level of ambiguity is on average lower than the level of implied risk, but it is more volatile. The sum of implied risk and ambiguity measures the total variance of the market return. It has a very high correlation (0.9448) with the VIX square, but they are not exactly the same. The reason is that the implied risk and ambiguity in this section is based on the option contracts with the shortest possible maturity date²⁰, while the VIX Index yields a weighted maturity of 30 days. This difference in the maturity date explains why the VIX square is on average larger than the sum of risk and ambiguity.

Next, I focus on the three recessions during this period. All three of them have higher average levels of total variance, measured either by the sum of implied risk and ambiguity or the VIX square, than of the whole period. The levels of implied risk during recessions are larger than during the whole period. The recessions in 2008 and 2020 exhibit significantly higher levels of ambiguity compared to the whole period, while the recession in 2001 has a similar ambiguity level as usual.

In fact, the increases of the total variance during these three recessions are driven by different sources. For the 2001 recession, the level of implied risk dominates the level of ambiguity. I view it as a recession driven mainly by risk. The 2020 recession, on the contrary, is mainly driven by ambiguity, since the average level of ambiguity is larger than that of implied risk. The 2008 recession has more balanced implied risk and ambiguity whose average levels are quite similar.

Furthermore, the standard deviations of implied risk and ambiguity exhibit different patterns during these three recessions. During the 2001 recession, the standard deviation of implied risk is larger than that of ambiguity. The fluctuations in the total variance are mainly driven by implied risk, according to the correlations among implied risk, ambiguity, and their sum. Differently, the 2020 recession has a more volatile ambiguity than implied risk, and the total variance co-moves more with

²⁰ The maturity dates of these options are always smaller than 30 days.

ambiguity instead of risk. As for the 2008 recession, the standard deviations of implied risk and ambiguity are close to each other, and they together drive the movement of total variance.

During the whole period, the correlation between implied risk and ambiguity is 0.6013. It goes up to 0.7004 in the 2008 recession, indicating that the levels of implied risk and ambiguity tend to move together during this period. On the contrary, the 2001 and 2020 recessions have relatively small correlations between implied risk and ambiguity, which are 0.3657 and 0.3862, respectively.

Let us look at the latter two recessions in Figure 1. The recent recession due to COVID-19 is of a very short period. In fact, the level of ambiguity increases dramatically to a very high value during March 12th-16th, and then decreases relatively quickly to a low level compared with implied risk. A similar pattern happens in the 2008 recession, where the ambiguity level decreases quickly after the peak but the implied risk remains at a high level for a longer period. Comparing the 2020 recession to the 2008 recession, it seems that the investors really had no clue about what happened when the pandemic suddenly bursted out but quickly returned to normal.

To view these results more clearly, I smooth the curves of implied risk and ambiguity by calculating their monthly averages. For each month between January 1996 and December 2020, I make an average of daily levels of implied risk and ambiguity within this month, and plot them in Figure 3. The black solid line is the monthly level of ambiguity, while the red dashed line is the monthly level of implied risk. This monthly figure exhibits results consistent with those in the daily figure: first, the three recessions are driven by different factors; second, the level of ambiguity decreases faster than that of implied risk after reaching the peak in both the 2008 recession and the 2020 recession. I further calculate the fraction of ambiguity in the sum of risk and ambiguity, and plot it as the black dash-dotted line in Figure 3. This fraction has already increased and stayed at a high level before the bursting of the 2008 Financial Crisis. In fact, the fraction remains a high-than-average level²¹ for 23 months starting from July 2007, which is also the month when the credit spread²² spikes up. The average level of this fraction becomes larger after 2015 compared with previous years.

²¹ The average level of this fraction during the whole period is 0.3144

²² The credit spread is measured by the difference between the interest rates on interbank loans and short-term U.S. government debt.

	Min.	Max.	Median	Mean	Std. dev.	Cross-correlations			
<i>Whole Period (January 1996 - December 2020):</i>									
Implied Risk	0.0013	0.4291	0.017	0.0240	0.0272	1			
Ambiguity	1.61E−6	1.1133	0.0055	0.0132	0.0310	0.6013	1		
Implied Risk + Ambiguity	0.0032	1.1888	0.0251	0.0372	0.0521	0.8797	0.9089	1	
VIX square	0.0084	0.6838	0.0347	0.0486	0.0525	0.8897	0.8074	0.9448	1
<i>Dot-com Bubble (March 2001 - November 2001):</i>									
Implied Risk	0.0144	0.1013	0.0373	0.0406	0.0156	1			
Ambiguity	1.88E−5	0.0632	0.0097	0.0129	0.0120	0.3657	1		
Implied Risk + Ambiguity	0.0231	0.1501	0.0458	0.0535	0.0229	0.8730	0.7732	1	
VIX square	0.0352	0.1913	0.0608	0.0719	0.0300	0.8637	0.7432	0.9780	1
<i>2008 Financial Crisis (December 2007 - June 2009):</i>									
Implied Risk	0.0029	0.4291	0.0419	0.0604	0.0563	1			
Ambiguity	1.48E−5	0.3392	0.0308	0.0515	0.0585	0.7004	1		
Implied Risk + Ambiguity	0.0210	0.7320	0.0664	0.1118	0.1058	0.9189	0.9252	1	
VIX square	0.0266	0.6538	0.0829	0.1364	0.1206	0.8645	0.9075	0.9613	1
<i>COVID-19 Crash (February 2020 - April 2020):</i>									
Implied Risk	0.0037	0.3916	0.0616	0.1083	0.1110	1			
Ambiguity	0.0013	1.1133	0.0570	0.1269	0.1967	0.3862	1		
Implied Risk + Ambiguity	0.0081	1.1888	0.1374	0.2352	0.2605	0.7177	0.9195	1	
VIX square	0.0187	0.6838	0.1638	0.1987	0.1645	0.8032	0.8202	0.9615	1

Table 1: Summary Statistics. This table presents the minimum, maximum, median, mean, standard deviation, and cross-correlations of implied risk, ambiguity, implied risk plus ambiguity, and the VIX square. They are calculated over the whole period (from January 1996 to December 2020) and the three recessions in the United States within this period according to NBER: the dot-com bubble (from March 2001 to November 2001), the financial crisis (from December 2007 to June 2009), and the COVID-19 pandemic (from February 2020 to April 2020).

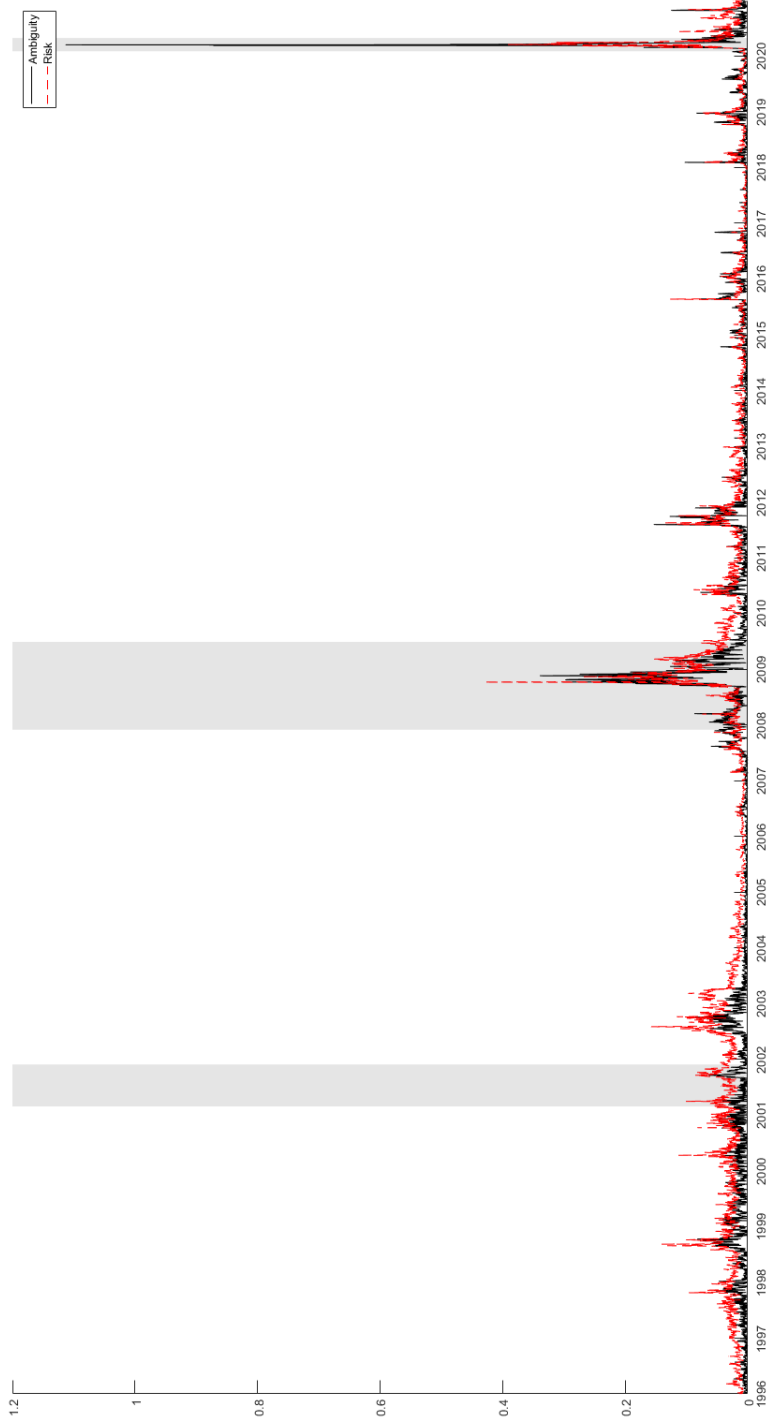


Figure 1: Daily levels of ambiguity and implied risk from January 1996 to December 2020. The black solid line is the level of ambiguity, while the red dashed line is the level of implied risk. They are obtained from the asymptotic expansion for the implied volatility. The shaded areas are three recessions in the United States during 1996-2020 according to NBER: the dot-com bubble (from March 2001 to November 2001), the financial crisis (from December 2007 to June 2009), and the COVID-19 pandemic (from February 2020 to April 2020).

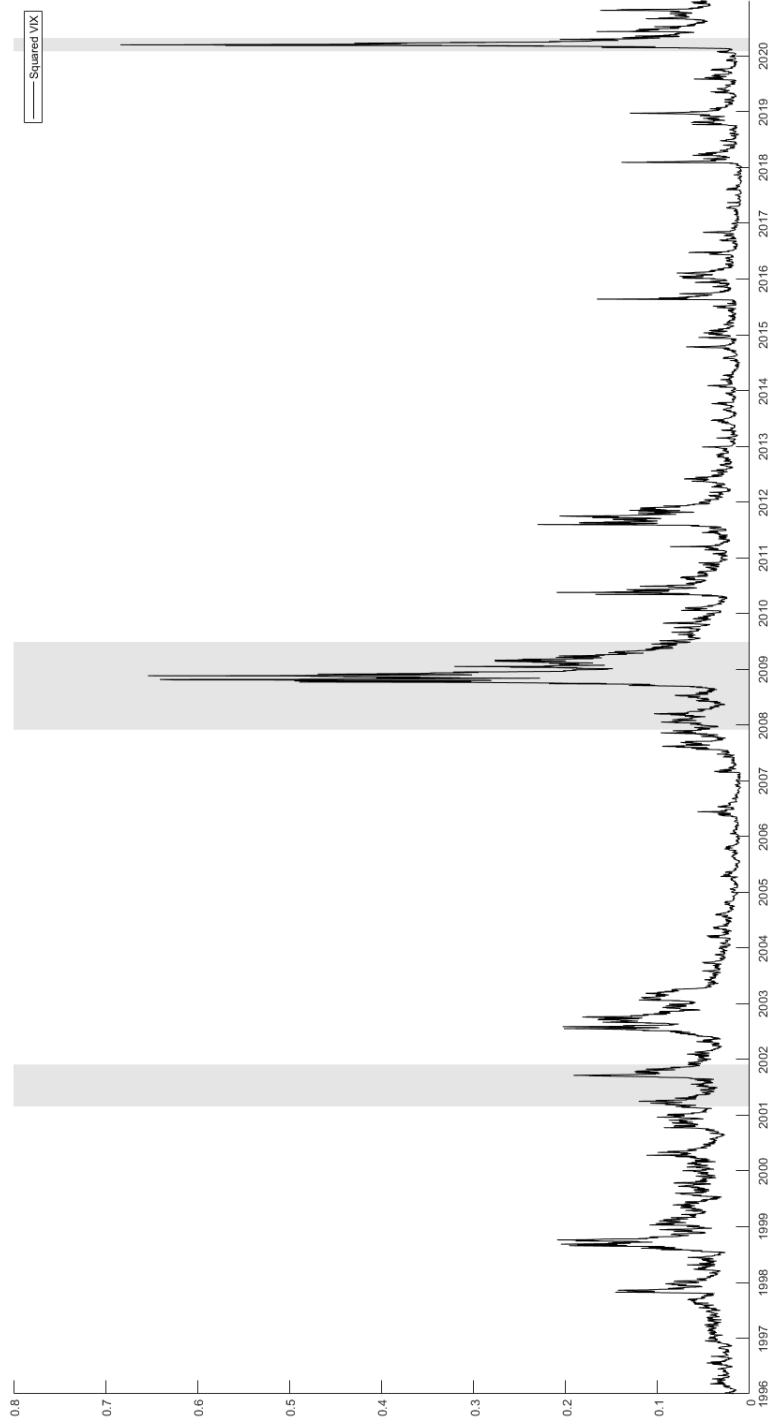


Figure 2: Daily VIX square from January 1996 to December 2020. The black solid line is the VIX square. The shaded areas are three recessions in the United States during 1996-2020 according to NBER: the dot-com bubble (from March 2001 to November 2001), the financial crisis (from December 2007 to June 2009), and the COVID-19 pandemic (from February 2020 to April 2020).

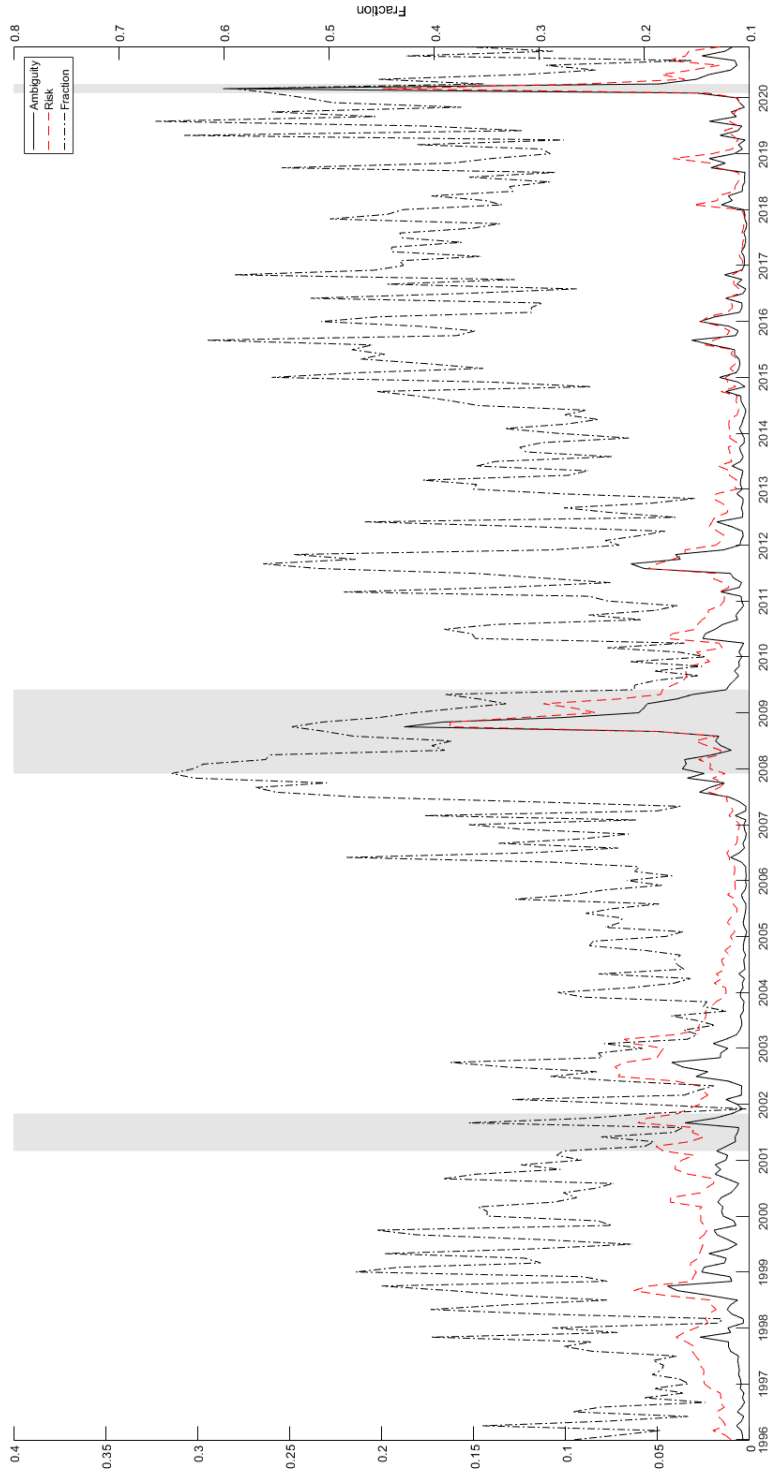


Figure 3: Monthly levels of ambiguity, implied risk, and the fraction of ambiguity in their sum from January 1996 to December 2020. The black solid line is the monthly level of ambiguity, while the red dashed line is the monthly level of implied risk. They are obtained by averaging the daily levels of ambiguity and implied risk within a month. The black dash-dotted line is the fraction of ambiguity in the sum of ambiguity and implied risk. The shaded areas are three recessions in the United States during 1996-2020 according to NBER: the dot-com bubble (from March 2001 to November 2001), the financial crisis (from December 2007 to June 2009), and the COVID-19 pandemic (from February 2020 to April 2020).

4.2 Results Based on Option Pricing Formula

Results based on asymptotic expansion for implied volatility offers insight into the evolution of the ambiguity level from 1996 to 2020. Now I turn to the formal estimation strategy which is based on the option pricing formula in Proposition 4.

For the reduced-form model (15)-(18) in Section 2.2, Σ_2 is not identifiable and needs to be normalized. Here I normalize it to be -1 to guarantee that the correlation between the risky asset's return and its variance is negative. This negative correlation captures the leverage effect (Christie, 1982). After the normalization, the parameter vector of the reduced-form model is denoted by $\Xi = (\Lambda_0, \Lambda_1; \zeta_0, \zeta_1; \Delta_0, \Delta_1; \Sigma_1; \sigma)'$.

To quantify the level of ambiguity, I modify the two-step calibration method in Christoffersen et al. (2009). This method treats the ambiguity level as another parameter that is estimated daily. Thus, both the parameter vector and the ambiguity level are estimated together with option data in an iterative two-step procedure as follows.

- *Step 1.* For a given parameter vector $\hat{\Xi}$, solve the error minimization problem for each day $t = 1, \dots, d$,

$$\hat{\kappa}_t = \arg \min_{\kappa_t} \frac{1}{N_t} \sum_{j=1}^{N_t} (IV_{j,t} - IV_j(\hat{\Xi}, \kappa_t))^2;$$

- *Step 2.* For a given set of $\{\hat{\kappa}_t\}_{t=1}^d$ obtained in Step 1, solve the aggregate error minimization problem

$$\hat{\Xi} = \arg \min_{\Xi} \frac{1}{N} \sum_{t=1}^d \sum_{j=1}^{N_t} (IV_{j,t} - IV_j(\Xi, \hat{\kappa}_t))^2;$$

where

- N_t is the number of option contracts on day t ;
- N is the aggregate number of option contracts from day 1 to day d , which is equal to $\sum_{t=1}^d N_t$;
- $IV_{j,t}$ is the implied volatility of option contract j on day t , which comes from the market data;

- $IV_j(\Xi, \kappa_t)$ is the model-based implied volatility of option contract j on day t , which depends on both the parameter vector and the ambiguity level.

The procedure iterates between *Step 1* and *Step 2* until there is no further significant decrease in the objective function value of *Step 2*.

I first set the initial values of both the parameter vector and the ambiguity level between day 1 and day d . For each option in the data set, I calculate its model-based price using the pricing formula in Proposition 4 given the parameter vector and the current level of ambiguity. Then, I apply the Black-Scholes formula to changing this model-based price to its implied volatility, and compare it with that from the market data. For each day, I find out the level of ambiguity that makes the model-based implied volatilities of option contracts on that day as close as possible to the market-based ones. For all option contracts between day 1 and day d , I solve the aggregate error minimization problem of implied volatilities to obtain the optimal parameter vector. In other words, the parameter vector is fixed from day 1 to day d , while the level of ambiguity is estimated daily.

I apply the above method to the S&P 500 Index options processed in Section 3. Here I mainly focus on the recent two recessions in the United States, which are the 2008-2009 financial crisis and the stock market crash due to COVID-19. The results based on asymptotic expansion in 4.1 show that both the changes and the average levels of ambiguity and implied risk are larger than usual during these two periods of crisis. I will compare the results obtained by the two-step calibration method to the asymptotic results.

Before presenting the results based on the two-step calibration method, I will discuss a few more settings for the estimation procedure. The first one is about the number of options used in the estimation. On every trading day there are a lot of option contracts with different strike prices and maturity dates, and this amount becomes larger and larger over time. Doing the estimation with all of these options causes a heavy burden of computational cost in practice. To decrease this burden, I reduce the number of option contracts used in the estimation while at the same time guaranteeing that the "selected" options are representative for the whole data set. Here I make use of the interpolated implied volatility curves obtained in Section 3. On each trading day and for a given maturity date, there is an interpolated curve of implied volatilities with respect to strike prices. I divide the range of available strike prices into n data points²³ and obtain their implied volatilities from the interpolated

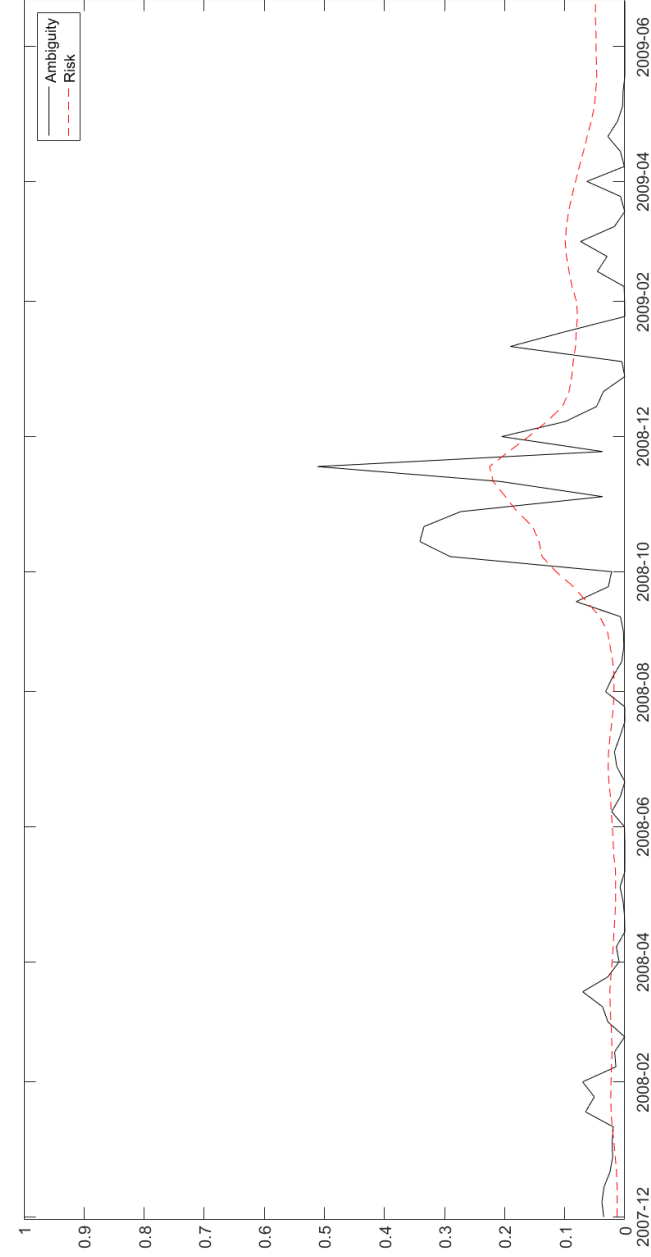
²³ I let n to be 5 in the later estimation. In fact, the choice of n does not influence the estimation

curve. It turns out that these pairs of strike prices and implied volatilities represent the whole curve well. I will use them in the later estimation instead of the original option data. The second setting is about the trading dates used in the estimation. In principle, I can estimate the level of ambiguity for each trading day. However, to further decrease the computational burden, I only use option data of each Wednesday to do the estimation. This choice follows Bates (2000) and Christoffersen et al. (2009). The third one is about the choice of d for *Step 1*. I assume d to be 4 in this section. This assumption indicates that: the parameter vector keeps same for every Wednesday in 4 weeks, while the level of ambiguity changes every Wednesday. The last setting is about the rolling-window technique used in the estimation. I choose the increment between two successive rolling windows to be 1 week (i.e. 1 Wednesday based on the above settings). Due to this choice, for each trading day in the data set, I get 4 estimates of the ambiguity level and 4 estimates of the implied risk level²⁴. I take an average of them respectively, and plot them in Figure 4. The black solid line is the level of ambiguity, while the red dashed line is the level of implied risk. Figure 4a represents the 2008-2009 financial crisis, while Figure 4b plots the recent stock market crash due to COVID-19.

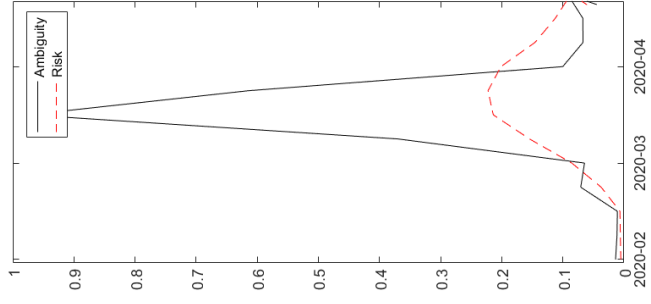
From Figure 4, we can see that the levels of ambiguity and implied risk show similar patterns as those in Figure 1 and 3 during the recent two recessions. In the 2008 recession, the average levels of ambiguity and implied risk are 0.0474 and 0.0613 respectively, while in the 2020 recession they are 0.1925 and 0.1120. These four numbers obtained from the formal estimation results are slightly different from those based on the asymptotic expansion of implied volatility, since they are estimated using different option data sets. Yet, the patterns keep unchanged that the 2020 recession is mainly driven by ambiguity while the 2008 recession is driven by both ambiguity and implied risk. Furthermore, the level of ambiguity fluctuates more than the level of implied risk. During both recessions, the level of ambiguity decreases to the normal level quickly after reaching peaks, while the level of implied risk remains at a high level for a longer period.

results a lot.

²⁴ These estimates are turned out to be quite stable for this data set



(a) 2007-2009



(b) 2020

Figure 4: Levels of ambiguity and implied risk during the 2008-2009 financial crisis and the COVID-19 pandemic. The black solid line is the level of ambiguity, while the red dashed line is the level of implied risk. (a) is the 2008-2009 financial crisis, which ranges from December 2007 to June 2009. (b) is the COVID-19 pandemic, which ranges from February 2020 to April 2020.

5 Conclusion

In this paper, I quantify the level of ambiguity in the US stock market using option data. This quantification of the ambiguity level is based on a reduced-form model of the market index which has two diffusion terms representing risk and ambiguity respectively. The presence of ambiguity as a diffusion term is not exogenously assumed, but based on a representative agent asset pricing model where the representative investor is averse to both risk and ambiguity. Ambiguity is modelled using the framework of recursive multiple-priors utility. This model serves as an economic foundation for the reduced-form model which keeps its main characteristics and is easy to estimate. Based on this reduced-form model, I derive a closed-form option pricing formula of the market index. I also provide an asymptotic expansion for the corresponding short-maturity at-the-money implied volatility. This approximation shows how the relative levels of risk and ambiguity affect the at-the-money level and slope of the implied volatility curve.

Using both the asymptotic expansion and the two-step calibration method based on the option pricing formula, I quantify the level of ambiguity with option data from 1996 to 2020 which covers the 2008-2009 financial crisis and the recent COVID-19 stock market crash. I find that both the level of ambiguity and the level of implied risk are much higher than usual during these two crises, and the level of ambiguity decreases to its normal level more rapidly than risk after spikes. Furthermore, the role of ambiguity is more pronounced for the recent stock market crash due to COVID-19 than for the 2008 financial crisis.

Appendix A

Proof of Proposition 1

Suppose the ex-dividend price of the risky asset under reference measure P_0 follows

$$dS_t = (r_t + \lambda_t - \frac{D_t}{S_t})S_t dt + \sigma_{1,t}S_t dB_{1,t} + \sigma_{2,t}S_t dB_{2,t},$$

where λ_t is the excess return and $\frac{D_t}{S_t}$ is the dividend yield.

Since ω_t is the fraction of wealth W_t invested in the risky asset, the dynamics of W_t is

$$\begin{aligned} dW_t &= \omega_t W_t \frac{dS_t + D_t dt}{S_t} + (1 - \omega_t)W_t r_t dt - C_t dt \\ &= [\omega_t W_t \lambda_t + W_t r_t - C_t]dt + \omega_t \sigma_{1,t} W_t dB_{1,t} + \omega_t \sigma_{2,t} W_t dB_{2,t} \end{aligned} \quad (22)$$

For the utility optimization problem (4), I need to solve the HJB equation given in (5):

$$\sup_{(C, \omega)} \mathcal{D}J(W, \kappa) + f(C, J(W, \kappa)) - \kappa \sigma_{J1} = 0,$$

where

$$\begin{aligned} \mathcal{D}J(W, \kappa) &= \frac{\partial J}{\partial \kappa} a(\bar{\kappa} - \kappa) + \frac{\partial J}{\partial W} [\omega W \lambda + W r - C] + \frac{1}{2} \frac{\partial^2 J}{\partial \kappa^2} b^2 \kappa \\ &\quad + \frac{1}{2} \frac{\partial^2 J}{\partial W^2} \omega^2 W^2 (\sigma_1^2 + \sigma_2^2) + \frac{\partial^2 J}{\partial W \partial \kappa} \omega W \sigma_2 b \sqrt{\kappa}. \end{aligned}$$

Taking the derivatives with respect to C and ω , I obtain the following first-order conditions:

$$\frac{\partial J}{\partial W} = \frac{\partial f}{\partial C} = \frac{\beta C^{\rho-1}}{(\alpha J)^{(\rho-\alpha)/\alpha}}, \quad (23)$$

$$\frac{\partial J}{\partial W} W \lambda + \frac{\partial^2 J}{\partial W^2} W^2 (\sigma_1^2 + \sigma_2^2) \omega + \frac{\partial^2 J}{\partial W \partial \kappa} W \sigma_2 b \sqrt{\kappa} - \kappa \frac{\partial \sigma_{J1}}{\partial \omega} = 0. \quad (24)$$

Conjecture that

$$J(W, \kappa) = \exp\left(\frac{\alpha(1-\rho)}{\rho} h(\kappa)\right) \frac{W^\alpha}{\alpha}.$$

and

$$\sigma_{J1} = \omega \alpha J \sigma_1$$

From (23), I get the wealth-consumption ratio , which is

$$\frac{W}{C} = \beta^{1/(\rho-1)} \exp(h(\kappa)). \quad (25)$$

The fraction of wealth invested in the risky asset is

$$\omega = \frac{\lambda + \frac{\alpha(1-\rho)}{\rho} h'(\kappa) \sigma_2 b \sqrt{\kappa} - \kappa \sigma_1}{(1-\alpha)(\sigma_1^2 + \sigma_2^2)}, \quad (26)$$

which comes from (24).

The HJB equation (5) is still hold, which is

$$\begin{aligned} & \frac{1-\rho}{\rho} h'(\kappa) a(\bar{\kappa} - \kappa) + [r + \omega \lambda - \beta^{1/(1-\rho)} \exp(-h(\kappa))] + \frac{1-\rho}{2\rho} [h''(\kappa) + \frac{\alpha(1-\rho)}{\rho} h'(\kappa)^2] b^2 \kappa \\ & - \frac{1}{2} (1-\alpha) \omega^2 (\sigma_1^2 + \sigma_2^2) + \frac{\alpha(1-\rho)}{\rho} h'(\kappa) \omega \sigma_2 b \sqrt{\kappa} + \frac{1}{\rho} \beta^{1/(1-\rho)} \exp(-h(\kappa)) - \frac{\beta}{\rho} - \kappa \omega \sigma_1 = 0. \end{aligned} \quad (27)$$

In this economy, market clearing implies the representative investor consumes exactly the aggregate dividends given by (3), so that $C^* = D$. Together with the wealth-consumption ratio, the wealth process in equilibrium is

$$\begin{aligned} dW_t = & [\mu_D + h'(\kappa_t) a(\bar{\kappa} - \kappa_t) + \frac{1}{2} (h'(\kappa_t)^2 + h''(\kappa_t)) b^2 \kappa_t] W_t dt \\ & + \sigma_D W_t dB_{1,t} + h'(\kappa_t) b \sqrt{\kappa_t} W_t dB_{2,t}. \end{aligned} \quad (28)$$

Furthermore, in equilibrium of a representative-agent model, there is zero net supply of the risk-free asset, i.e. $\omega^* = 1$. Therefore, in equilibrium I have:

$$\sigma_{1,t} = \sigma_D, \quad (29)$$

$$\sigma_{2,t} = h'(\kappa_t) b \sqrt{\kappa_t}. \quad (30)$$

The return of the risky asset in equilibrium is

$$\lambda_t + r_t = \mu_D + h'(\kappa_t) a(\bar{\kappa} - \kappa_t) + \frac{1}{2} (h'(\kappa_t)^2 + h''(\kappa_t)) b^2 \kappa_t + \beta^{1/(1-\rho)} \exp(-h(\kappa_t)), \quad (31)$$

where according to (26) the excess return λ_t is

$$\lambda_t = (1-\alpha) \sigma_D^2 + \kappa_t \sigma_D + \frac{\rho-\alpha}{\rho} h'(\kappa_t)^2 b^2 \kappa_t, \quad (32)$$

and the risk-free rate is

$$r_t = \mu_D - (1-\alpha)\sigma_D^2 - \kappa_t\sigma_D + h'(\kappa_t)a(\bar{\kappa} - \kappa_t) + \frac{1}{2}\left(\frac{2\alpha - \rho}{\rho}h'(\kappa_t)^2 + h''(\kappa_t)\right)b^2\kappa_t + \beta^{1/(1-\rho)}\exp(-h(\kappa_t)). \quad (33)$$

Now, (27) turns into

$$0 = \mu_D - \frac{\beta}{\rho} - \frac{1}{2}(1-\alpha)\sigma_D^2 - \kappa_t\sigma_D + \frac{1}{\rho}h'(\kappa_t)a(\bar{\kappa} - \kappa_t) + \frac{1}{2\rho}h''(\kappa_t)b^2\kappa_t + \frac{\alpha}{2\rho^2}h'(\kappa_t)^2b^2\kappa_t + \frac{1}{\rho}\beta^{1/(1-\rho)}\exp(-h(\kappa_t)), \quad (34)$$

and thus

$$\beta^{1/(1-\rho)}\exp(-h(\kappa_t)) = \beta + \frac{1}{2}\rho(1-\alpha)\sigma_D^2 - \rho\mu_D + \rho\kappa_t\sigma_D - h'(\kappa_t)a(\bar{\kappa} - \kappa_t) - \frac{\alpha}{2\rho}h'(\kappa_t)^2b^2\kappa_t - \frac{1}{2}h''(\kappa_t)b^2\kappa_t. \quad (35)$$

Substituting it into (33), I have

$$r_t = \beta + (1-\rho)\mu_D - (1-\rho)\kappa_t\sigma_D - \frac{1}{2}(1-\alpha)(2-\rho)\sigma_D^2 - \frac{\rho-\alpha}{2\rho}h'(\kappa_t)^2b^2\kappa_t. \quad (36)$$

In this economy, the equilibrium investment portfolio is solely made up of the risky asset and all dividends are used for consumption in equilibrium. Under these two conditions, the dividend yield in equilibrium is $\frac{D}{S} = \frac{C}{W} = \beta^{1/(1-\rho)}\exp(-h(\kappa))$, which is equal to

$$\frac{D}{S} = \beta + \frac{1}{2}\rho(1-\alpha)\sigma_D^2 - \rho\mu_D + \rho\kappa_t\sigma_D - h'(\kappa_t)a(\bar{\kappa} - \kappa_t) - \frac{\alpha}{2\rho}h'(\kappa_t)^2b^2\kappa_t - \frac{1}{2}h''(\kappa_t)b^2\kappa_t, \quad (37)$$

according to (35).

To get an analytical solution, I use the log-linear approximation proposed by Collin-Dufresne and Goldstein (2005) and also used by Shi (2019) that $h(\kappa_t)$ is approximated by an affine function of κ_t , which is written as $h(\kappa_t) \approx \eta_0 + \eta_1\kappa_t$.

According to Collin-Dufresne and Goldstein (2005) and Shi (2019), this approximation should essentially minimize the expected squared approximation error of (34). Thus, the parameters η_0 and η_1 can be obtained by solving the following optimization problem:

$$\min_{\eta_0, \eta_1} E\left(\left[\mu_D - \frac{\beta}{\rho} - \frac{1}{2}(1-\alpha)\sigma_D^2 - \kappa_t\sigma_D + \frac{1}{\rho}\eta_1a(\bar{\kappa} - \kappa_t) + \frac{\alpha}{2\rho^2}\eta_1^2b^2\kappa_t + \frac{1}{\rho}\beta^{1/(1-\rho)}\exp(-\eta_0 - \eta_1\kappa_t)\right]^2\right). \quad (38)$$

This optimization problem is equivalent to the one in page 640 of Shi (2019).²⁵ Shi (2019) also studies the accuracy of this approximation in his Appendix B, and claims that this approximation performs fairly well.

With the approximation of $h(\kappa_t)$ as $\hat{\eta}_0 + \hat{\eta}_1 \kappa_t$,²⁶ now I have:

$$\sigma_{1,t} = \sigma_D, \quad (39)$$

$$\sigma_{2,t} = \hat{\eta}_1 b \sqrt{\kappa_t}, \quad (40)$$

$$r_t = \left[\beta + (1 - \rho)\mu_D - \frac{1}{2}(1 - \alpha)(2 - \rho)\sigma_D^2 \right] - \left[(1 - \rho)\sigma_D + \frac{\rho - \alpha}{2\rho} \hat{\eta}_1^2 b^2 \right] \kappa_t. \quad (41)$$

$$\lambda_t = (1 - \alpha)\sigma_D^2 + \kappa_t \sigma_D + \frac{\rho - \alpha}{\rho} \hat{\eta}_1^2 b^2 \kappa_t, \quad (42)$$

$$\frac{D_t}{S_t} = \left[\beta + \frac{1}{2}\rho(1 - \alpha)\sigma_D^2 - \rho\mu_D - \hat{\eta}_1 a \bar{\kappa} \right] + \left[\rho\sigma_D + \hat{\eta}_1 a - \frac{\alpha}{2\rho} \hat{\eta}_1^2 b^2 \right] \kappa_t \quad (43)$$

Therefore, the price of the risky asset under reference measure P_0 follows

$$\frac{dS_t}{S_t} = (r_t + \lambda_t - \frac{D_t}{S_t})dt + \sigma_c dB_{1,t} + \hat{\eta}_1 b \sqrt{\kappa_t} dB_{2,t}, \quad (44)$$

where the risk-free rate r_t , the excess return λ_t , and the dividend yield $\frac{D_t}{S_t}$ are given by (41), (42), and (43) respectively. Furthermore, κ_t follows (2) under reference measure P_0 .

Proof of Proposition 2

According to Lemma 5.1 of Chen and Epstein (2002), the pricing kernel M_t under reference measure P_0 should satisfy

$$M_t = \exp \left(\int_0^t \frac{\partial f}{\partial J}(C_s, J_s) ds \right) \frac{\partial f}{\partial C}(C_t, J_t) z_t^*,$$

where z_t^* is the Radon-Nikodym derivative between the worst-case model and the reference model.

²⁵ I can also write all terms in (38) except for the exponential term as an affine function of κ_t . The two coefficients are equal to $-\frac{1}{\alpha}$ times n_0 and n_1 in page 640 of Shi (2019) respectively. Specifically, σ_{ca} in n_1 is equal to 0 under the setting of this model.

²⁶ $\hat{\eta}_0$ and $\hat{\eta}_1$ come from the minimization of (38).

For the Epstein-Zin preference $f(C, J)$, its partial derivatives are:

$$\begin{aligned}\frac{\partial f}{\partial C} &= \frac{\beta C^{\rho-1}}{(\alpha J)^{(\rho-\alpha)/\alpha}}, \\ \frac{\partial f}{\partial J} &= -\frac{\beta\alpha}{\rho} - \beta \frac{\rho-\alpha}{\rho} \frac{C^\rho}{(\alpha J)^{\rho/\alpha}}.\end{aligned}$$

In equilibrium, since the representative investor consumes exactly the aggregate dividends, i.e. $C^* = D$, the indirect utility function J is a function of D and κ , which is written as $J(D, \kappa) = \beta^{\alpha/(\rho-1)} \exp\left(\frac{\alpha}{\rho} h(\kappa)\right) \frac{D^\alpha}{\alpha}$.

The natural logarithm of M_t is

$$\begin{aligned}\ln(M_t) &= - \int_0^t \left(\frac{\beta\alpha}{\rho} + \frac{\rho-\alpha}{\rho} \beta^{1/(1-\rho)} \exp(-h(\kappa_s)) \right) ds \\ &\quad + \frac{1-\alpha}{1-\rho} \ln(\beta) - (1-\alpha) \ln(D_t) - \frac{\rho-\alpha}{\rho} h(\kappa_t) + \ln(z_t^*),\end{aligned}$$

and I denote it as Y_t .

I apply the Ito's lemma to obtain the dynamics of Y_t :

$$\begin{aligned}dY_t &= - \left[\frac{\beta\alpha}{\rho} + \frac{\rho-\alpha}{\rho} \beta^{1/(1-\rho)} \exp(-h(\kappa_s)) \right] dt - (1-\alpha) \frac{dD_t}{D_t} - \frac{\rho-\alpha}{\rho} h'(\kappa_t) d\kappa_t + \frac{dz_t^*}{z_t^*} \\ &\quad + \frac{1}{2} (1-\alpha) \frac{1}{D_t^2} d\langle D_t, D_t \rangle - \frac{1}{2} \frac{\rho-\alpha}{\rho} h''(\kappa_t) d\langle \kappa_t, \kappa_t \rangle - \frac{1}{2} \frac{1}{(z_t^*)^2} d\langle z_t^*, z_t^* \rangle.\end{aligned}$$

The dynamics of κ_t and D_t are given by (2) and (3) respectively, and z_t^* is the Radon-Nikodym derivative between the worst-case model and the reference model. Therefore,

$$\begin{aligned}dY_t &= - \left[\frac{\beta\alpha}{\rho} + \frac{\rho-\alpha}{\rho} \beta^{1/(1-\rho)} \exp(-h(\kappa_s)) \right] dt - (1-\alpha) (\mu_D dt + \sigma_D dB_{1,t}) \\ &\quad - \frac{\rho-\alpha}{\rho} h'(\kappa_t) (a(\bar{\kappa} - \kappa_t) dt + b\sqrt{\kappa_t} dB_{2,t}) - \kappa_t dB_{1,t} \\ &\quad + \frac{1}{2} (1-\alpha) \sigma_D^2 dt - \frac{1}{2} \frac{\rho-\alpha}{\rho} h''(\kappa_t) b^2 \kappa_t dt - \frac{1}{2} \kappa_t^2 dt \\ &= - \left[\frac{\beta\alpha}{\rho} + \frac{\rho-\alpha}{\rho} \beta^{1/(1-\rho)} \exp(-h(\kappa_t)) + (1-\alpha) \mu_D + \frac{\rho-\alpha}{\rho} h'(\kappa_t) a(\bar{\kappa} - \kappa_t) \right. \\ &\quad \left. - \frac{1}{2} (1-\alpha) \sigma_D^2 + \frac{1}{2} \frac{\rho-\alpha}{\rho} h''(\kappa_t) b^2 \kappa_t + \frac{1}{2} \kappa_t^2 \right] dt \\ &\quad - [(1-\alpha) \sigma_D + \kappa_t] dB_{1,t} - \frac{\rho-\alpha}{\rho} h'(\kappa_t) b \sqrt{\kappa_t} dB_{2,t}.\end{aligned}$$

Substituting $\beta^{1/(1-\rho)} \exp(-h(\kappa_t))$ with (35), the above equation changes into

$$\begin{aligned} dY_t = & - \left[\beta + (1-\rho)\mu_D + \frac{1}{2}(\rho-\alpha-1)(1-\alpha)\sigma_D^2 + (\rho-\alpha)\kappa_t\sigma_D \right. \\ & \left. - \frac{\alpha(\rho-\alpha)}{2\rho^2}h'(\kappa_t)^2b^2\kappa_t + \frac{1}{2}\kappa_t^2 \right] dt \\ & - [(1-\alpha)\sigma_D + \kappa_t]dB_{1,t} - \frac{\rho-\alpha}{\rho}h'(\kappa_t)b\sqrt{\kappa_t}dB_{2,t}. \end{aligned}$$

To obtain the dynamics of M_t , I make use of the relation $M_t = \exp(Y_t)$ and apply Ito's lemma again:

$$dM_t = \exp(Y_t)dY_t + \frac{1}{2}\exp(Y_t)d\langle Y_t, Y_t \rangle = M_t dY_t + \frac{1}{2}M_t d\langle Y_t, Y_t \rangle.$$

Therefore,

$$\begin{aligned} \frac{dM_t}{M_t} = & - \left[\beta + (1-\rho)\mu_D + \frac{1}{2}(\rho-\alpha-1)(1-\alpha)\sigma_D^2 + (\rho-\alpha)\kappa_t\sigma_D \right. \\ & \left. - \frac{\alpha(\rho-\alpha)}{2\rho^2}h'(\kappa_t)^2b^2\kappa_t + \frac{1}{2}\kappa_t^2 \right] dt \\ & - [(1-\alpha)\sigma_D + \kappa_t]dB_{1,t} - \frac{\rho-\alpha}{\rho}h'(\kappa_t)b\sqrt{\kappa_t}dB_{2,t} \\ & + \frac{1}{2}[(1-\alpha)\sigma_D + \kappa_t]^2dt + \frac{1}{2}\left(\frac{\rho-\alpha}{\rho}\right)^2h'(\kappa_t)^2b^2\kappa_tdt \\ = & - \left[\beta + (1-\rho)\mu_D - (1-\rho)\kappa_t\sigma_D - \frac{1}{2}(1-\alpha)(2-\rho)\sigma_D^2 - \frac{\rho-\alpha}{2\rho}h'(\kappa_t)^2b^2\kappa_t \right] dt \\ & - [(1-\alpha)\sigma_D + \kappa_t]dB_{1,t} - \frac{\rho-\alpha}{\rho}h'(\kappa_t)b\sqrt{\kappa_t}dB_{2,t}. \end{aligned}$$

According to (36), the drift term of the above equation is equivalent to $-r_t dt$, where r_t is the risk-free rate in equilibrium. I further simplify it using the approximation of $h(\kappa_t)$, and thus obtain the dynamics of M_t as

$$\frac{dM_t}{M_t} = -r_t dt - [(1-\alpha)\sigma_D + \kappa_t]dB_{1,t} - \frac{\rho-\alpha}{\rho}\hat{\eta}_1 b\sqrt{\kappa_t}dB_{2,t}, \quad (45)$$

where the risk-free rate r_t is given by (41).²⁷

Proof of Proposition 3

Given the pricing kernel and the price of the risky asset under physical measure P_0 , now I derive the price of the risky asset under risk-neutral measure Q .

²⁷ Shi (2019) provides a similar result about the pricing kernel in his Equation (14).

Suppose $\tilde{B}_t = \begin{pmatrix} \tilde{B}_{1,t} \\ \tilde{B}_{2,t} \end{pmatrix}$ is a Brownian motion under risk-neutral measure Q . Since the dynamics of the pricing kernel is given by (45), I use the following equation to switch from measure P_0 to measure Q according to the Girsanov theorem:

$$\begin{pmatrix} d\tilde{B}_{1,t} \\ d\tilde{B}_{2,t} \end{pmatrix} = \begin{pmatrix} dB_{1,t} \\ dB_{2,t} \end{pmatrix} + \begin{pmatrix} (1-\alpha)\sigma_D + \kappa_t \\ \frac{\rho-\alpha}{\rho}\hat{\eta}_1 b\sqrt{\kappa_t} \end{pmatrix} dt. \quad (46)$$

Applying it to (44), I get the price of the risky asset under risk-neutral measure Q as

$$\frac{dS_t}{S_t} = \left(r_t - \frac{D_t}{S_t}\right)dt + \sigma_D d\tilde{B}_{1,t} + \hat{\eta}_1 b\sqrt{\kappa_t} d\tilde{B}_{2,t}, \quad (47)$$

where r_t is the risk-free rate given by (41) and $\frac{D_t}{S_t}$ is the dividend yield given by (43).

Furthermore, the dynamics of κ_t under risk-neutral measure Q is

$$d\kappa_t = \left[a\bar{\kappa} - \left(a + \frac{\rho-\alpha}{\rho}\hat{\eta}_1 b^2\right)\kappa_t\right]dt + b\sqrt{\kappa_t}d\tilde{B}_{2,t}, \quad (48)$$

where r_t is the risk-free rate given by (41).

In this way, I obtain the dynamics of the risky asset's price under risk-neutral measure Q , which can be used for pricing European options based on this underlying asset.

Appendix B

Consider the reduced-form model (15) - (18) in Section 2.2. In this appendix, I derive an expression for

$$\psi(u; T, S_0, \kappa_0) = E \left[\exp \left(- \int_0^T r_s ds \right) e^{u \cdot \ln(S_T)} \middle| S_0, \kappa_0 \right],$$

which will be used to obtain the option pricing formula in Appendix C and the characteristic function of $\ln(S_T)$ in Appendix D.

The derivation is based on Duffie et al. (2000) who provide a general solution to $\psi(u; T, S_0, \kappa_0)$ for affine diffusions. First of all, I verify that the above reduced-form model satisfies their requirement of affine diffusion state processes.

Denoting the state vector of this model $(\ln(S_t), \kappa_t)'$ by X_t , its dynamics is

$$dX_t = \left[\begin{pmatrix} \Lambda_0 - \zeta_0 - \frac{1}{2}\Sigma_1^2 \\ \Delta_0 \end{pmatrix} - \begin{pmatrix} 0 & \Lambda_1 + \zeta_1 + \frac{1}{2}\Sigma_2^2 \\ 0 & \Delta_1 \end{pmatrix} X_t \right] dt + \begin{pmatrix} \Sigma_1 & \Sigma_2\sqrt{\kappa_t} \\ 0 & \sigma\sqrt{\kappa_t} \end{pmatrix} \begin{pmatrix} d\tilde{B}_{1,t} \\ d\tilde{B}_{2,t} \end{pmatrix}, \quad (49)$$

and the risk-free rate is $r_t = \Lambda_0 - \Lambda_1\kappa_t$.

The state vector X_t indeed follows an affine diffusion process, since:

- the drift term is $\mu(X_t) = K_0 + K_1X_t$, where

$$K_0 = \begin{pmatrix} \Lambda_0 - \zeta_0 - \frac{1}{2}\Sigma_1^2 \\ \Delta_0 \end{pmatrix} \quad \text{and} \quad K_1 = - \begin{pmatrix} 0 & \Lambda_1 + \zeta_1 + \frac{1}{2}\Sigma_2^2 \\ 0 & \Delta_1 \end{pmatrix};$$

- the volatility term is

$$\sigma(X_t)\sigma(X_t)^T = \begin{pmatrix} \Sigma_1^2 + (0, \Sigma_2^2)X_t & (0, \Sigma_2\sigma)X_t \\ (0, \Sigma_2\sigma)X_t & (0, \sigma^2)X_t \end{pmatrix},$$

$$\text{where } \sigma(X_t) = \begin{pmatrix} \Sigma_1 & \Sigma_2\sqrt{\kappa_t} \\ 0 & \sigma\sqrt{\kappa_t} \end{pmatrix}.$$

Furthermore, the discount rate $R(X_t) = r_t$ is also an affine function of the state vector X_t , since $R(X_t) = R_0 + R_1X_t$ with $R_0 = \Lambda_0$ and $R_1 = - \begin{pmatrix} 0 & \Lambda_1 \end{pmatrix}$.

Therefore, the reduced-form model fulfills the requirement of Duffie et al. (2000) and I can apply their transform analysis method to the derivation of $\psi(u; T, S_0, \kappa_0)$.²⁸

²⁸ In fact, the function $\psi(u; T, S_0, \kappa_0)$ is a special case of Equation (2.3) in Duffie et al. (2000), which

According to Equation (2.4) in Duffie et al. (2000), the function $\psi(u; T, S_0, \kappa_0)$ is given by

$$\psi(u; T, S_0, \kappa_0) = \exp(\bar{\alpha}(u; 0, T) + \bar{\beta}_1(u; 0, T) \ln(S_0) + \bar{\beta}_2(u; 0, T) \kappa_0). \quad (50)$$

We can figure out $\bar{\alpha}(u; 0, T)$, $\bar{\beta}_1(u; 0, T)$, and $\bar{\beta}_2(u; 0, T)$ from the following ordinary differential equations (ODEs)²⁹:

$$\frac{d\bar{\beta}_1(u; t, T)}{dt} = 0, \quad (51)$$

$$\begin{aligned} \frac{d\bar{\beta}_2(u; t, T)}{dt} = & -\Lambda_1 + (\Lambda_1 + \zeta_1 + \frac{1}{2}\Sigma_2^2)\bar{\beta}_1(u; t, T) + \Delta_1\bar{\beta}_2(u; t, T) - \frac{1}{2}\Sigma_2^2\bar{\beta}_1(u; t, T)^2 \\ & - \Sigma_2\sigma\bar{\beta}_1(u; t, T)\bar{\beta}_2(u; t, T) - \frac{1}{2}\sigma^2\bar{\beta}_2(u; t, T)^2, \end{aligned} \quad (52)$$

$$\frac{d\bar{\alpha}(u; t, T)}{dt} = \Lambda_0 - (\Lambda_0 - \zeta_0 - \frac{1}{2}\Sigma_1^2)\bar{\beta}_1(u; t, T) - \Delta_0\bar{\beta}_2(u; t, T) - \frac{1}{2}\Sigma_1^2\bar{\beta}_1(u; t, T)^2. \quad (53)$$

The boundary conditions at $t = T$ are: $\bar{\beta}_1(u; T, T) = u$, $\bar{\beta}_2(u; T, T) = 0$, and $\bar{\alpha}(u; T, T) = 0$.

Based on (51), $\bar{\beta}_1(u; t, T)$ is a constant and always equal to u . In other words, for any t ,

$$\bar{\beta}_1(u; t, T) = u \quad (54)$$

Plugging it into (52), we have

$$\begin{aligned} \frac{d\bar{\beta}_2(u; t, T)}{dt} = & [-\Lambda_1 + (\Lambda_1 + \zeta_1 + \frac{1}{2}\Sigma_2^2)u - \frac{1}{2}\Sigma_2^2u^2] + [\Delta_1 - \Sigma_2\sigma u]\bar{\beta}_2(u; t, T) \\ & - \frac{1}{2}\sigma^2\bar{\beta}_2(u; t, T)^2. \end{aligned} \quad (55)$$

Letting

$$\begin{cases} \Gamma_0(u) = -2\Lambda_1 + 2(\Lambda_1 + \zeta_1 + \frac{1}{2}\Sigma_2^2)u - \Sigma_2^2u^2 \\ \Gamma_1(u) = \Sigma_2\sigma u - \Delta_1 \\ \xi(u) = \sqrt{\Gamma_1(u)^2 + \Gamma_0(u)\sigma^2} \end{cases},$$

is equal to $\psi(u_1, u_2; T, S_0, \kappa_0) = E\left[\exp\left(-\int_0^T r_s ds\right)e^{u_1 \cdot \ln(S_T) + u_2 \cdot \kappa_0} | S_0, \kappa_0\right]$ in the terminology of the above reduced-form model. u_2 is always equal to 0 in the derivation of both the option pricing formula and the characteristic function of $\ln(S_T)$

²⁹ See Duffie et al. (2000)'s Equation (2.5) and (2.6) for these ODEs.

(55) is simplified as

$$\frac{d\bar{\beta}_2(u; t, T)}{dt} = \frac{1}{2}\Gamma_0(u) - \Gamma_1(u)\bar{\beta}_2(u; t, T) - \frac{1}{2}\sigma^2\bar{\beta}_2(u; t, T)^2. \quad (56)$$

This ODE is of "separation of variables" type whose general solution is known. Together with the boundary condition $\bar{\beta}_2(u; T, T) = 0$, we can obtain the expression of $\bar{\beta}_2(u; t, T)$ as follows:

$$\bar{\beta}_2(u; t, T) = -\frac{\Gamma_1(u)}{\sigma^2} + \frac{\xi(u)}{\sigma^2} \cdot \frac{(\xi(u) + \Gamma_1(u))e^{-\xi(u)(T-t)} - (\xi(u) - \Gamma_1(u))}{(\xi(u) + \Gamma_1(u))e^{-\xi(u)(T-t)} + (\xi(u) - \Gamma_1(u))}. \quad (57)$$

With $\bar{\beta}_1(u; t, T)$ and $\bar{\beta}_2(u; t, T)$ solved, we can figure out $\bar{\alpha}(u; t, T)$ by integrating from t to T and using the boundary condition $\bar{\alpha}(u; T, T) = 0$. The expression of $\bar{\alpha}(u; t, T)$ is

$$\begin{aligned} \bar{\alpha}(u; t, T) = & -\left[\Lambda_0 - \left(\Lambda_0 - \zeta_0 - \frac{1}{2}\Sigma_1^2\right)u - \frac{1}{2}\Sigma_1^2u^2\right](T-t) \\ & - \Delta_0\left[\frac{\xi(u) + \Gamma_1(u)}{\sigma^2}(T-t) + \frac{2}{\sigma^2}\ln\left(1 - \frac{\xi(u) + \Gamma_1(u)}{2\xi(u)}(1 - e^{-\xi(u)(T-t)})\right)\right]. \end{aligned} \quad (58)$$

Letting $t = 0$ in $\bar{\beta}_1(u; t, T)$, $\bar{\beta}_2(u; t, T)$, and $\bar{\alpha}(u; t, T)$, I obtain the expression of $\psi(u; T, S_0, \kappa_0)$. This result will be used later to derive the option pricing formula in Appendix C and the characteristic function of $\ln(S_T)$ in Appendix D.

Appendix C

Proof of Proposition 4

Consider the reduced-form model (15) - (18) in Section 2.2. The price of an European call option on the risky asset with strike price K and maturity data T at $t = 0$ is

$$C_0 = E \left[\exp \left(- \int_0^T r_s ds \right) (S_T - K)^+ | S_0, \kappa_0 \right] = G_1(K, T, S_0, \kappa_0) - K \cdot G_2(K, T, S_0, \kappa_0), \quad (59)$$

where

$$G_1(K, T, S_0, \kappa_0) = \frac{1}{2} \psi(1; T, S_0, \kappa_0) + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re} \left(\frac{K^{-i\phi} \psi(1 + i\phi; T, S_0, \kappa_0)}{i\phi} \right) d\phi, \quad (60)$$

$$G_2(K, T, S_0, \kappa_0) = \frac{1}{2} \psi(0; T, S_0, \kappa_0) + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re} \left(\frac{K^{-i\phi} \psi(i\phi; T, S_0, \kappa_0)}{i\phi} \right) d\phi, \quad (61)$$

according to Duffie et al. (2000).³⁰

The function form of $\psi(u; T, S_0, \kappa_0)$ has already been solved out in Appendix B. It is determined by (50), (54), (57), and (58):

$$\psi(u; T, S_0, \kappa_0) = \exp \left(\bar{\alpha}(u; 0, T) + \bar{\beta}_1(u; 0, T) \ln(S_0) + \bar{\beta}_2(u; 0, T) \kappa_0 \right),$$

where

$$\begin{aligned} \bar{\beta}_1(u; 0, T) &= u, \\ \bar{\beta}_2(u; 0, T) &= -\frac{\Gamma_1(u)}{\sigma^2} + \frac{\xi(u)}{\sigma^2} \cdot \frac{(\xi(u) + \Gamma_1(u))e^{-\xi(u)T} - (\xi(u) - \Gamma_1(u))}{(\xi(u) + \Gamma_1(u))e^{-\xi(u)T} + (\xi(u) - \Gamma_1(u))}, \\ \bar{\alpha}(u; 0, T) &= -\left[\Lambda_0 - (\Lambda_0 - \zeta_0 - \frac{1}{2}\Sigma_1^2)u - \frac{1}{2}\Sigma_1^2 u^2 \right] T \\ &\quad - \Delta_0 \left[\frac{\xi(u) + \Gamma_1(u)}{\sigma^2} T + \frac{2}{\sigma^2} \ln \left(1 - \frac{(\xi(u) + \Gamma_1(u))}{2\xi(u)} (1 - e^{-\xi(u)T}) \right) \right], \end{aligned}$$

and

$$\begin{cases} \Gamma_0(u) = -2\Lambda_1 + 2(\Lambda_1 + \zeta_1 + \frac{1}{2}\Sigma_2^2)u - \Sigma_2^2 u^2 \\ \Gamma_1(u) = \Sigma_2 \sigma u - \Delta_1 \\ \xi(u) = \sqrt{\Gamma_1(u)^2 + \Gamma_0(u)\sigma^2} \end{cases}.$$

Letting u equal to 0, 1, $i\phi$, and $1 + i\phi$ respectively, I obtain the call option pricing

³⁰ See Equation (2.9) and (2.12) in Duffie et al. (2000).

formula of the risky asset.

Appendix D

Consider a special case of the reduced-form model (15) - (18) in Section 2.2 where $\Lambda_0 = 0$, $\Lambda_1 = 0$, $\zeta_0 = 0$, $\zeta_1 = 0$, and $\Sigma_2 = -1$.

Let $IV(m)$ denote the function of implied volatility with respect to moneyness m at time $t = 0$. When time to maturity $T \rightarrow 0$, the asymptotic expansion of $IV(m)$ around $m = 0$ is

$$IV(m) = \sqrt{\Sigma_1^2 + \kappa_0} - \frac{1}{4} \frac{\sigma \kappa_0}{(\sqrt{\Sigma_1^2 + \kappa_0})^3} m + \left[\frac{1}{24} \frac{\sigma^2}{(\sqrt{\Sigma_1^2 + \kappa_0})^3} - \frac{5}{48} \frac{\sigma^2 \kappa_0^2}{(\sqrt{\Sigma_1^2 + \kappa_0})^7} \right] m^2 + O(m^3). \quad (62)$$

This approximation is derived based on Jacquier and Lorig (2015), which provide an explicit implied volatility approximation for any model with an analytically tractable characteristic function.

The characteristic function of $\ln(S_T)$ in this case is $\Phi(\phi) = E[e^{i\phi \ln(S_T)} | S_0, \kappa_0]$. It is equal to $\psi(u; T, S_0, \kappa_0)$ in Appendix B when $\Lambda_0 = 0$, $\Lambda_1 = 0$, $\zeta_0 = 0$, $\zeta_1 = 0$, $\Sigma_2 = -1$, and $u = i\phi$. Plugging these parameter values into (54), (57), and (58), I have

$$\begin{aligned} \bar{\beta}_1(i\phi; 0, T) &= i\phi, \\ \bar{\beta}_2(i\phi; 0, T) &= -\frac{\Gamma_1(i\phi)}{\sigma^2} + \frac{\xi(i\phi)}{\sigma^2} \frac{(\xi(i\phi) + \Gamma_1(i\phi))e^{-\xi(i\phi)T} - (\xi(i\phi) - \Gamma_1(i\phi))}{(\xi(i\phi) + \Gamma_1(i\phi))e^{-\xi(i\phi)T} + (\xi(i\phi) - \Gamma_1(i\phi))}, \\ \bar{\alpha}(i\phi; 0, T) &= -\frac{1}{2}\Sigma_1^2(\phi^2 + i\phi)T - \Delta_0 \left[\frac{1}{\sigma^2} (\xi(i\phi) + \Gamma_1(i\phi))T \right. \\ &\quad \left. + \frac{2}{\sigma^2} \ln \left(1 - \frac{\xi(i\phi) + \Gamma_1(i\phi)}{2\xi(i\phi)} (1 - e^{-\xi(i\phi)T}) \right) \right], \end{aligned}$$

where

$$\begin{cases} \Gamma_0(i\phi) &= i\phi + \phi^2, \\ \Gamma_1(i\phi) &= -i\phi\sigma - \Delta_1, \\ \xi(i\phi) &= \sqrt{\Gamma_1(i\phi)^2 + \Gamma_0(i\phi)\sigma^2} \end{cases}.$$

Therefore, the characteristic function of $\ln(S_T)$ is

$$\Phi(\phi) = \psi(i\phi; T, S_0, \kappa_0) = \exp \left(\bar{\alpha}(i\phi; 0, T) + \bar{\beta}_1(i\phi; 0, T) \ln(S_0) + \bar{\beta}_2(i\phi; 0, T) \kappa_0 \right)$$

Letting $\bar{\alpha}(i\phi; 0, T) + \bar{\beta}_2(i\phi; 0, T) \kappa_0$ denote by $\varphi(T, \phi)$, this characteristic function $\Phi(\phi)$ turns to be

$$\Phi(\phi) = \exp \left(i\phi \ln(S_0) + \varphi(T, \phi) \right).$$

For models with this kind of characteristic function³¹, I can derive the asymptotic expansion for implied volatility using the results in Jacquier and Lorig (2015).

According to Jacquier and Lorig (2015), the characteristic function of $\ln(S_T)$ in the Black-Scholes model is also needed to derive the expansion for implied volatility, since the Black-Scholes formula serves as the intermediate for transforming option price to implied volatility. The characteristic function of $\ln(S_T)$ in the Black-Scholes model is:

$$\Phi_0(\phi) = \exp(i\phi \ln(S_0) + \varphi_0(\phi; \sigma_0)), \quad (63)$$

where

$$\varphi_0(\phi; \sigma_0) = -\frac{1}{2}(\phi^2 + i\phi)\sigma_0^2. \quad (64)$$

σ_0 is the volatility in the Black-Scholes model.

I define

$$\varphi^\varepsilon(T, \phi; \sigma_0) \equiv T\varphi_0(\phi; \sigma_0) + \varepsilon\varphi_1(T, \phi; \sigma_0),$$

where

$$\varphi_1(T, \phi; \sigma_0) = \varphi(T, \phi) - T\varphi_0(\phi; \sigma_0).$$

$\varphi^\varepsilon(T, \phi; \sigma_0)$ can be viewed as the perturbation around the Black-Scholes model. When $\varepsilon = 0$, $\varphi^\varepsilon(T, \phi; \sigma_0) = T\varphi_0(\phi; \sigma_0)$; when $\varepsilon = 1$, $\varphi^\varepsilon(T, \phi; \sigma_0) = \varphi(T, \phi)$.

According to Jacquier and Lorig (2015), the derivation of the expansion for implied volatility needs two auxiliary functions. The first one is

$$a_k(T; \sigma_0) = \frac{(-i)^k}{k!} \partial_\phi^k \varphi_1(T, \phi; \sigma_0) \Big|_{\phi=0},$$

which is Equation (11) in Jacquier and Lorig (2015). The second one is the physicist's Hermite polynomial of n th order:

$$H_n(z) = (-1)^n e^{z^2} \partial_z^n e^{-z^2},$$

which is mentioned in Theorem 2 of Jacquier and Lorig (2015).

I make use of the approximation formula for implied volatility given in Example 1 of Jacquier and Lorig (2015), which is

$$\sigma^{(2,3)} = \sigma_0 + \sigma_1^{(3)} + \sigma_2^{(3)},$$

³¹ This characteristic function is included in case (iii) of Equation (2) in Jacquier and Lorig (2015).

where

$$\begin{aligned}\sigma_1^{(3)} &= \frac{u_1^{(3)}}{\partial_\sigma u^{BS}} \\ \sigma_2^{(3)} &= \frac{u_2^{(3)}}{\partial_\sigma u^{BS}} - \frac{1}{2}(\sigma_1^{(3)})^2 \frac{\partial_\sigma^2 u^{BS}}{\partial_\sigma u^{BS}}.\end{aligned}$$

Letting $z_0 = -\frac{1}{\sigma_0\sqrt{2T}}(m + \frac{1}{2}\sigma_0^2T)$, we have

$$\begin{aligned}\frac{u_1^{(3)}}{\partial_\sigma u^{BS}} &= \frac{1}{\sigma_0 T} \left[\left(\frac{-1}{\sigma_0\sqrt{2T}} \right) a_3 H_1(z_0) + (a_2 + a_3) \right], \\ \frac{u_2^{(3)}}{\partial_\sigma u^{BS}} &= \frac{1}{\sigma_0 T} \left[\left(\frac{-1}{\sigma_0\sqrt{2T}} \right)^4 a_3^2 H_4(z_0) + \left(\frac{-1}{\sigma_0\sqrt{2T}} \right)^3 (2a_2 a_3 + a_3^2) H_3(z_0) \right. \\ &\quad \left. + \left(\frac{-1}{\sigma_0\sqrt{2T}} \right)^2 (a_2^2 - a_3^2) H_2(z_0) + \left(\frac{-1}{\sigma_0\sqrt{2T}} \right) (-a_2^2 - a_3^2 - 2a_2 a_3) H_1(z_0) \right], \\ \frac{\partial_\sigma^2 u^{BS}}{\partial_\sigma u^{BS}} &= \frac{1}{\sigma_0 T} \left[\sigma_0^2 T^2 \left(\frac{-1}{\sigma_0\sqrt{2T}} \right)^2 H_2(z_0) - \sigma_0^2 T^2 \left(\frac{-1}{\sigma_0\sqrt{2T}} \right) H_1(z_0) + T \right],\end{aligned}$$

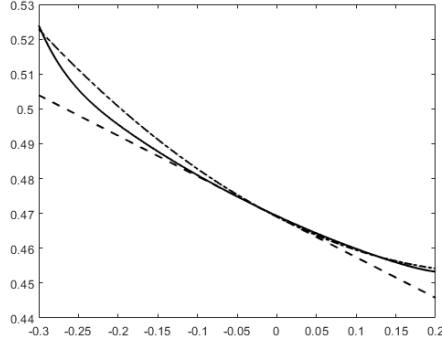
where a_k is the abbreviation of $a_k(T; \sigma_0)$.

The above terms can be computed with the help of Mathematica. After that, I figure out the limit of this approximation when $T \rightarrow 0$ using L'Hospital's rule. The final result is presented in Equation (62).

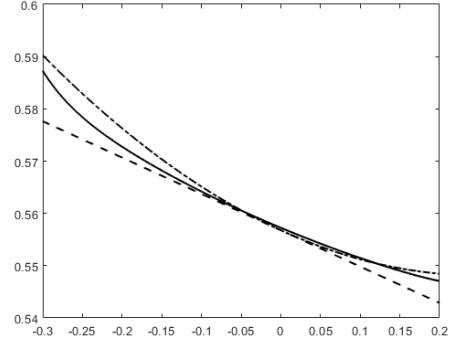
Next, I will show the performance of this approximation by comparing it with the numerically obtained implied volatility. Figure 5 plots the implied volatility with respect to the logarithm of moneyness for different parameter settings. The solid line corresponds to the implied volatility calculated numerically from the option price. The dashed line corresponds to the first-order approximation for the implied volatility, which is the first two terms of Equation (62). The dash-dotted line corresponds to the second-order approximation for the implied volatility, which is Equation (62) without the residual term. The baseline setting of parameters is as follows: $\Sigma_1 = 0.4$, $\sigma = 0.8$, $\Delta_1 = 5$, $\frac{\Delta_0}{\Delta_1} = 0.08$, and $\kappa_0 = 0.06$. The maturity time is set to be 0.04, and keeps same for different parameter settings in Figure 5. The baseline setting is plotted in Figure 5a. Then I change the setting of each parameter separately while keeping other parameters unchanged. The results are shown in Figure 5b - 5f. From Figure 5, we can see that the performance of the approximation given in Equation (62) is quite good especially around the at-the-money level.

Figure 6 studies the effect of maturity time on this approximation, with x-axis being time to maturity. Figure 6a corresponds to the value of the implied volatility

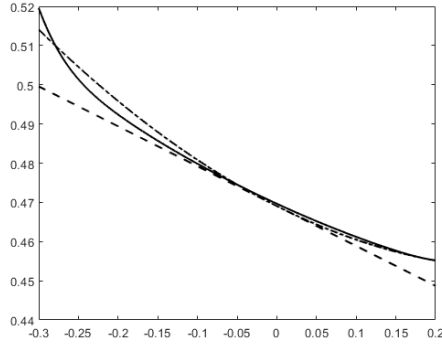
at $m = 0$. Figure 6b and 6c correspond to the 1st and 2nd order derivatives of the implied volatility curve with respect to the logarithm of moneyness when $m = 0$. The solid lines are obtained by first calculating the implied volatility numerically and then computing the 1st and 2nd derivatives numerically. The dashed lines are coefficients of Equation (62) respectively, which do not depend on maturity time. Since the approximation given in Equation (6) holds when time to maturity goes to zero, I would expect that the solid lines become closer to the dashed lines with maturity time being smaller. Figure 6 is consistent with this hypothesis.



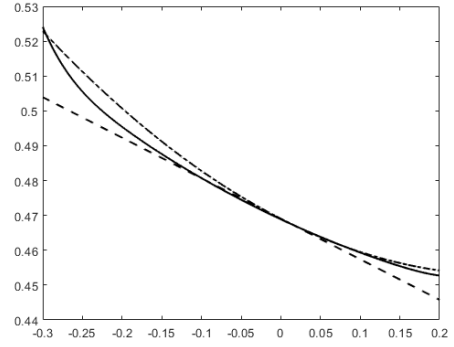
(a) Baseline



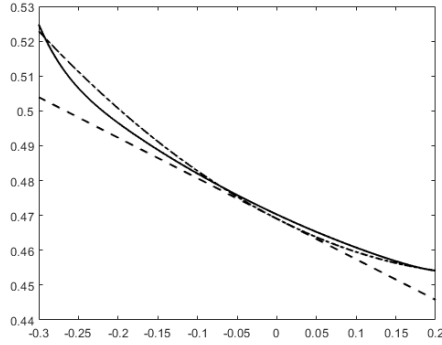
(b) $\Sigma_1 = 0.5$



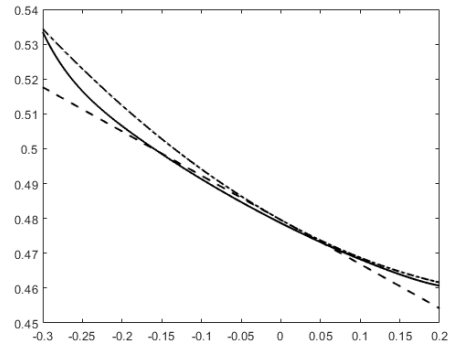
(c) $\sigma = 0.7$



(d) $\Delta_1 = 4$

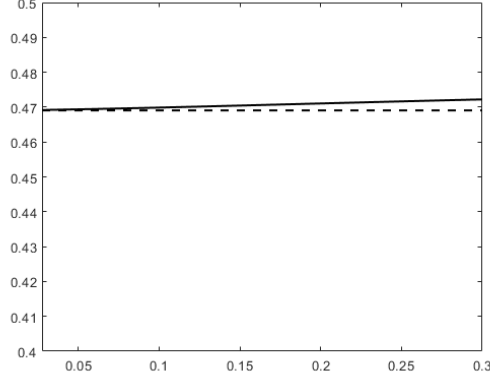


(e) $\frac{\Delta_0}{\Delta_1} = 0.09$

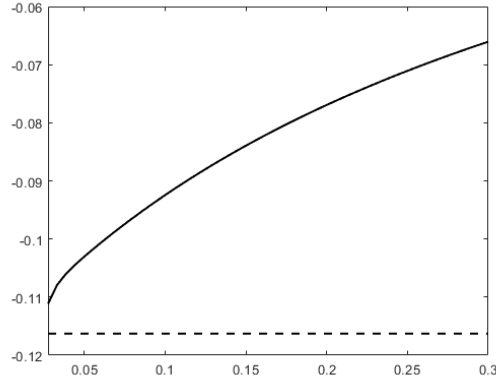


(f) $\kappa_0 = 0.07$

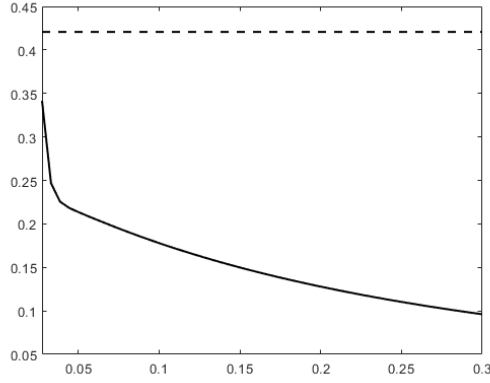
Figure 5: Implied volatility with respect to the logarithm of moneyness for different parameter settings. The solid line corresponds to the implied volatility calculated numerically from the option price. The dashed line corresponds to the first-order approximation for the implied volatility, which is the first two terms of Equation (62). The dash-dotted line corresponds to the second-order approximation for the implied volatility, which is Equation (62) without the residual term. The maturity time is set to be 0.04. The baseline setting of parameters is: $\Sigma_1 = 0.4$, $\sigma = 0.8$, $\Delta_1 = 5$, $\frac{\Delta_0}{\Delta_1} = 0.08$, and $\kappa_0 = 0.06$. Each parameter is changed separately while keeping other parameters same: (b) $\Sigma_1 = 0.5$; (c) $\sigma = 0.7$; (d) $\Delta_1 = 4$; (e) $\frac{\Delta_0}{\Delta_1} = 0.09$ (f) $\kappa_0 = 0.07$.



(a) $IV(0)$



(b) $IV'(0)$



(c) $IV''(0)$

Figure 6: Effect of maturity time on the approximation. The x-axis is time to maturity. (a) is the implied volatility at $m = 0$; (b) is the 1st order derivative of the implied volatility curve with respect to the logarithm of moneyness when $m = 0$; (c) is the 2nd order derivative of the implied volatility curve with respect to the logarithm of moneyness when $m = 0$. The solid lines are obtained by first calculating the implied volatility numerically and then computing the 1st and 2nd derivatives numerically. The dashed lines are $IV(0)$, $IV'(0)$, and $IV''(0)$ of Equation (62) respectively, which do not depend on maturity time.

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